

DYNKIN DIAGRAMS AND SHORT PEIRCE GRADINGS OF KANTOR PAIRS

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ABSTRACT. In a recent article with Oleg Smirnov, we defined short Peirce (SP) graded Kantor pairs. For any such pair P , we defined a family, parameterized by the Weyl group of type BC_2 , consisting of SP-graded Kantor pairs called Weyl images of P . In this article, we classify finite dimensional simple SP-graded Kantor pairs over an algebraically closed field of characteristic 0 in terms of marked Dynkin diagrams, and we show how to compute Weyl images using these diagrams. The theory is particularly attractive for close-to-Jordan Kantor pairs (which are variations of Freudenthal triple systems), and we construct the reflections of such pairs (with nontrivial gradings) starting from Jordan matrix pairs.

Suppose in this introduction that \mathbb{K} is a field of characteristic $\neq 2$ or 3 . A *Kantor pair* is a pair $P = (P^-, P^+)$ of \mathbb{K} -modules together with trilinear products $\{, , \}^\sigma : P^\sigma \times P^{-\sigma} \times P^\sigma \rightarrow P^\sigma$, $\sigma = \pm$, satisfying two 5-linear identities (see Section 4.2) which were first written down by Isai Kantor in [K1] in the special case of *Kantor triple systems* (when $P^- = P^+$ and $\{, , \}^- = \{, , \}^+$). Examples of Kantor pairs, or structures that give rise to Kantor pairs, have arisen in the work of many different authors (see Section 4.3 and [AFS, §3.1] for some references).

The motivation for the study of Kantor pairs is their relationship with 5-graded Lie algebras. We now recall that relationship in the special case of primary interest to us when the structures involved are simple, in which case the relationship is a 1-1 correspondence. (Our convention, as described in Subsection 2.1, is that the term *simple* for a graded structure is interpreted in the ungraded sense.) If $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ is a simple 5-graded Lie algebra, then (L_{-1}, L_1) is a simple Kantor pair with products given by $\{x, y, z\}^\sigma = [[x, y], z]$ for $\sigma = \pm$, in which case we say that the pair (L_{-1}, L_1) is *enveloped* by L . Conversely, any simple Kantor pair P is enveloped by a simple 5-graded Lie algebra L , which is unique up to graded isomorphism [AFS] and which has the property that $L_{\sigma 2} \simeq K^\sigma(P^\sigma, P^\sigma)$ (as vector spaces) for $\sigma = \pm$, where $K^\sigma(x, z) \in \text{Hom}(P^{-\sigma}, P^\sigma)$ is defined by $K^\sigma(x, z)w = \{x, w, z\}^\sigma - \{z, w, x\}^\sigma$ for $x, z \in P^\sigma$.

A Kantor pair P satisfying $K^\sigma(P^\sigma, P^\sigma) = 0$ for $\sigma = \pm$ is called a (linear) *Jordan pair* [L], and the relationship between Kantor pairs and 5-graded Lie algebras, generalizes the well-known relationship between Jordan pairs and 3-graded Lie algebras. With this important special case in mind, we define a *close-to-Jordan pair* to be a Kantor pair P with $\dim(K^\sigma(P^\sigma, P^\sigma)) = 1$ for $\sigma = \pm$.

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If Δ is a finite root system (possibly not reduced), we define a Δ -grading of a Lie algebra L , to be a grading of L by the root lattice of Δ , with support contained in $\Delta \cup \{0\}$ (see Section 3). With this terminology we see that 5-graded Lie algebras can be thought of as BC_1 -graded Lie algebras, by which we mean a Lie algebra graded by the root system $\{-2\alpha_1, -\alpha_1, \alpha_1, 2\alpha_1\}$ of type BC_1 . Hence the correspondence described above can be viewed as a 1-1 correspondence between simple Kantor pairs (up to isomorphism) and simple BC_1 -graded Lie algebras (up to graded isomorphism).

We next recall how BC_2 -graded Lie algebras also play a role in the study of Kantor pairs. A *short Peirce grading* (*SP-grading*) of a Kantor pair P is a \mathbb{Z} -grading $P = P_0 \oplus P_1$ of P with $P_i = 0$ for $i \neq 0, 1$. These gradings were introduced in [AFS], where it was shown that there is a 1-1 correspondence between simple SP-graded Kantor pairs and simple BC_2 -graded Lie algebras that is analogous to the one just described. Given a simple SP-graded Kantor pair P , this correspondence was used to construct a simple SP-graded Kantor pair uP , called the *u-image* of P , for each u in the Weyl group of the root system of type BC_2 . In this way one obtains eight *Weyl images* of a simple SP-graded Kantor pair P , no two of which are (in general) graded-isomorphic. Of particular interest is the s_1 -image of P , which we simply call the *reflection* of P , where s_1 is the reflection corresponding to the short basic root. As an application of Weyl images, it was shown in [AFS] that reflection applied to Jordan pairs of skew-transformations yields a new class of (in general) infinite dimensional simple SP-graded Kantor pairs that are not themselves Jordan.

In this article, we use subsets of Dynkin diagrams to classify simple SP-graded Kantor pairs and to compute their Weyl images in the special case of finite dimensional pairs over an algebraically closed field \mathbb{K} of characteristic 0. For the rest of this introduction, we make these additional assumptions and briefly outline our main results. In these results, a simple Kantor pair is said to be of *type* X_n , if the root system of its enveloping simple 5-graded Lie algebra is of type X_n .

With the exception of Section 5, which we discuss below, Sections 1 to 6 are devoted to laying the necessary groundwork on homomorphisms of root systems, root graded Lie algebras and Kantor pairs. Of particular interest is Theorem 3.2.2, which uses homomorphisms of root systems to classify all Δ -gradings (with Δ arbitrary) of a finite dimensional semi-simple Lie algebra.

In Section 7, we assume that Π is the Dynkin diagram of type X_n , where X_n is the type of an irreducible reduced finite root system. We first define a set $SPA(\Pi)$ of pairs of subsets of Π , whose elements are said to be *SP-admissible*. In Proposition 7.1.5. we show how to easily write down the elements of $SPA(\Pi)$ using the extended diagram of Π . Then, in Theorem 7.2.4, we give a classification of simple SP-graded Kantor pairs of type X_n by showing that they are (up to graded isomorphism) in 1-1 correspondence with the orbits in $SPA(\Pi)$ under the right action of $\text{Aut}(\Pi)$. The proof uses Theorem 3.2.2 when Δ is irreducible of type BC_2 .

If we specialize Theorem 7.2.4 to the case when the SP-grading is the zero grading (i.e. $P = P^0$), we obtain a classification of simple (ungraded) Kantor pairs of type X_n , which states that they are (up to isomorphism) in 1-1 correspondence with the orbits in the set $KA(\Pi)$ of *Kantor admissible* subsets of Π under the right action of $\text{Aut}(\Pi)$. For the sake of readability, we actually state this classification theorem earlier as Theorem 5.2.4 in Section 5. We note that Theorem 5.2.4 is equivalent to Kantor's classification of non-polarized simple Kantor triple systems [K1] (see

Remark 5.2.5). Nevertheless we include a complete treatment of the topic here, since we are working in the context of pairs rather than triple systems, and since the article [K1] is unavailable to some readers and does not include all details.

Both Theorems 5.2.4 and 7.2.4 take particularly simple and attractive forms in the case of close-to-Jordan pairs, and for this reason we highlight this example throughout the paper. For example, we see in Theorems 5.4.1 and 5.5.2 that if $n \geq 2$ there is exactly one simple close-to-Jordan pair of type X_n and that this Kantor pair can be viewed as the signed double of a Freudenthal triple system [M1] with a modified product.

If Section 8, we introduce a left action, denoted by $*$, of the Weyl group W_Δ of the root system Δ of type BC_2 on the set $\text{SPA}(\Pi)$. In Theorem 8.3.1, we show that, if $u \in W_\Delta$, the u -image of the SP-graded Kantor pair corresponding to $(S, T) \in \text{SPA}(\Pi)$ is the SP-graded Kantor pair corresponding to $u*(S, T) \in \text{SPA}(\Pi)$. Then in Theorem 8.3.2, we show how to compute $u*(S, T)$ using only information about the Dynkin diagram Π . The proof of this result is rather intricate, involving the longest element of the Weyl group of certain subsets of Π . Combining Theorems 8.3.1 and 8.3.2 with our classification Theorem 7.2.4, one can easily compute the Weyl images of any simple SP-graded Kantor pair using only information about Π .

As an application of our results, we obtain in Section 9 a construction of the reflections of all simple nontrivially SP-graded close-to-Jordan pairs in the form $J \otimes U$, where J is a Jordan pair of matrices and U is a pair of two-dimensional spaces. This generalizes Kantor's construction of his remarkable Kantor triple system C_{55}^2 , since the double of that triple system is a reflection of the close-to-Jordan pair of type E_6 [AFS, §7].

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1. SOME SETS OF HOMOMORPHISMS BETWEEN ROOT SYSTEMS

In this section we investigate some sets of homomorphisms between roots systems.

1.1. Root systems. In this paper, a *root system* will mean a finite root system Δ in a finite dimensional real Euclidean space E_Δ as described for example in [B, VI, §1.1]. Δ is said to be *reduced* if $(2\Delta) \cap \Delta = \emptyset$. The *rank* n_Δ of Δ is the dimension of E_Δ . If Δ is irreducible and $n = n_\Delta$, the *type* of Δ is one of $A_n (n \geq 1)$, $B_n (n \geq 2)$, $C_n (n \geq 3)$, $D_n (n \geq 4)$, $E_n (n = 6, 7, 8)$, F_4 , G_2 , or $BC_n (n \geq 1)$, where the last case occurs if and only if Δ is not reduced [B, VI, §1.4, Prop. 14].

We use the notation $Q_\Delta := \text{span}_{\mathbb{Z}}(\Delta)$ for the *root lattice* of Δ , which is a free abelian group of rank n_Δ . The *automorphism group* of Δ , denoted by $\text{Aut}(\Delta)$, is the stabilizer of Δ in $\text{GL}(E_\Delta)$. Using the restriction map, we often identify $\text{Aut}(\Delta)$ with the stabilizer of Δ in $\text{Aut}(Q_\Delta)$. The *Weyl group* of Δ (contained in $\text{Aut}(\Delta)$), will be denoted by W_Δ . We denote the Euclidean product on E_Δ by (\cdot, \cdot) , and write $\langle \alpha, \beta \rangle = 2(\alpha, \beta)/(\beta, \beta)$ for $\alpha, \beta \in \Delta$. If Δ is irreducible, we denote the set of roots of minimum length (*short roots*) in Δ by Δ_{sh} .

If a base Π_Δ for the root system Δ is fixed, we let Δ^+ be the set of *positive roots relative to* Π_Δ . If $\alpha = \sum_{\gamma \in \Pi_\Delta} n_\gamma \gamma \in Q_\Delta$, where $n_\gamma \in \mathbb{Z}$, we define the *height* of α to be $\text{ht}_{\Pi_\Delta}(\alpha) := \sum_{\gamma \in \Pi_\Delta} n_\gamma$ and the *support* of α in Π_Δ to be $\text{supp}_{\Pi_\Delta}(\alpha) :=$

$\{\gamma \in \Pi_\Delta : n_\gamma \neq 0\}$. We denote the stabilizer of Π_Δ in $\text{Aut}(\Delta)$ by $\text{Aut}(\Pi_\Delta)$. If Δ is reduced, the set Π_Δ has the additional structure of a *Dynkin diagram*, in which case the restriction map identifies $\text{Aut}(\Pi_\Delta)$ with the automorphism group of the Dynkin diagram Π_Δ [B, VI, §4.2]. If $\Pi_\Delta = \{\alpha_i : i \in I\}$ is indexed by a finite set I , the $I \times I$ -matrix $C(\Pi_\Delta) = (\langle \alpha_i, \alpha_j \rangle)_{i,j \in I}$ is called the *Cartan matrix* of Π_Δ .

If \mathcal{G} is a finite dimensional simple Lie algebra over an algebraically closed field of characteristic 0 with Cartan subalgebra \mathcal{H} , the classical theory for such Lie algebras tells us that the root system $\Sigma(\mathcal{G}, \mathcal{H})$ of \mathcal{G} relative to \mathcal{H} is an irreducible reduced root system in the Euclidean space $E_{\Sigma(\mathcal{G}, \mathcal{H})} = \mathbb{R} \otimes_{\mathbb{Q}} \text{span}_{\mathbb{Q}} \Sigma(\mathcal{G}, \mathcal{H})$ [H, §8.5]. In that case, the *type* of \mathcal{G} is defined to be the type of $\Sigma(\mathcal{G}, \mathcal{H})$.

1.2. The sets $\text{Hom}(\Sigma, \Delta)$ and $\text{Hom}_p(\Sigma, \Delta)$. For the rest of the section, we assume that Δ is a root system of rank n_Δ with base Π_Δ , and Σ is a root system of rank n_Σ with base Π_Σ . Elements of Q_Δ will be normally denoted by α, β, \dots , while elements of Q_Σ will normally be denoted by μ, ν, \dots .

Let

$$\text{Hom}(\Sigma, \Delta) := \{\rho \in \text{Hom}(Q_\Sigma, Q_\Delta) : \rho(\Sigma) \subseteq \Delta \cup \{0\}\}.$$

We call elements of $\text{Hom}(\Sigma, \Delta)$ *homomorphisms of Σ into Δ* . We say that $\rho \in \text{Hom}(\Sigma, \Delta)$ is *positive (relative to Π_Σ and Π_Δ)* if $\rho(\Sigma^+) \subseteq \Delta^+ \cup \{0\}$ (or equivalently $\rho(\Pi_\Sigma) \subseteq \Delta^+ \cup \{0\}$). Let

$$\text{Hom}_p(\Sigma, \Delta) := \{\rho \in \text{Hom}(\Sigma, \Delta) : \rho \text{ is positive}\}.$$

1.3. Some actions by composition. Recall that if G and G' are abelian groups (written additively) then there is a *right action of $\text{Aut}(G)$ by composition* on the group $\text{Hom}(G, G')$, as well as a *left action by composition of $\text{Aut}(G')$ on $\text{Hom}(G, G')$* . These are defined respectively by

$$\rho \cdot \varphi := \rho \circ \varphi \quad \text{and} \quad \theta \cdot \rho := \theta \circ \rho$$

for $\varphi \in \text{Aut}(G)$, $\rho \in \text{Hom}(G, G')$ and $\theta \in \text{Aut}(G')$. Since these two actions commute, we can write expressions like $\theta \cdot \rho \cdot \varphi$ for $\varphi \in \text{Aut}(G)$, $\rho \in \text{Hom}(G, G')$ and $\theta \in \text{Aut}(G')$.

In particular, $\text{Aut}(Q_\Sigma)$ acts on the right by composition and $\text{Aut}(Q_\Delta)$ acts on the left by composition on $\text{Hom}(Q_\Sigma, Q_\Delta)$. Moreover, clearly

$$\text{Aut}(\Delta) \cdot \text{Hom}(\Sigma, \Delta) \cdot \text{Aut}(\Sigma) \subseteq \text{Hom}(\Sigma, \Delta).$$

Hence $\text{Aut}(\Sigma)$ (*resp.* $\text{Aut}(\Delta)$) *acts on the right (resp. left) on $\text{Hom}(\Sigma, \Delta)$ by composition*.

Lemma 1.3.1. *Each orbit in $\text{Hom}(\Sigma, \Delta)$ under the right action of W_Σ by composition contains a unique element of the set $\text{Hom}_p(\Sigma, \Delta)$.*

Proof. We will need some notation for the proof. First let $\overline{C(\Pi_\Sigma)} = \{\mu \in E_\Sigma : (\mu, \nu) \geq 0 \text{ for } \nu \in \Pi_\Sigma\}$ denote the closure of the fundamental Weyl chamber in E_Σ determined by Π_Σ . Also, if $\tau \in \text{Hom}(Q_\Sigma, \mathbb{Z})$, then (since Q_Σ is a lattice in E_Σ) there exists a unique $\tau^* \in E_\Sigma$ such that $(\tau^*, \mu) = \tau(\mu)$ for $\mu \in Q_\Sigma$. Since W_Σ preserves the form (\cdot, \cdot) we see that

$$(\tau \cdot w)^* = w^{-1} \tau^*$$

for $\tau \in \text{Hom}(Q_\Sigma, \mathbb{Z})$ and $w \in W_\Sigma$. Moreover, if $\tau \in \text{Hom}(Q_\Sigma, \mathbb{Z})$, then $\tau^* \in \overline{C(\Pi_\Sigma)}$ if and only if $\tau(\mu) \geq 0$ for $\mu \in \Sigma^+$.

To prove existence, suppose that $\rho \in \text{Hom}(\Sigma, \Delta)$. Since $\rho \in \text{Hom}(Q_\Sigma, Q_\Delta)$, there exists unique elements $\rho_1, \dots, \rho_{n_\Delta} \in \text{Hom}(Q_\Sigma, \mathbb{Z})$ such that $\rho(\mu) = \sum_{i=1}^{n_\Delta} \rho_i(\mu) \alpha_i$ for $\mu \in Q_\Sigma$, where $\Pi_\Delta = \{\alpha_1, \dots, \alpha_{n_\Delta}\}$. Then if $\mu \in \Sigma$, we have

$$\rho_i(\mu) \rho_j(\mu) \geq 0 \text{ for } \mu \in \Sigma \text{ and } 1 \leq i \neq j \leq n_\Delta, \quad (1)$$

since $\rho(\mu) \in \Sigma \cup \{0\}$. Let $\tilde{\rho} := \sum_{i=1}^{n_\Delta} \rho_i \in \text{Hom}(Q_\Sigma, \mathbb{Z})$. Then, since $\overline{C(\Pi_\Sigma)}$ is a fundamental domain for W_Σ [B, V, § 3.2, Thm. 1], we can choose $w \in W_\Sigma$ such that $w^{-1} \tilde{\rho}^* \in \overline{C(\Pi_\Sigma)}$. Hence $(\tilde{\rho} \cdot w)^* \in \overline{C(\Pi_\Sigma)}$, so $(\tilde{\rho} \cdot w)(\mu) \geq 0$ for $\mu \in \Sigma^+$. Therefore, $\sum_{i=1}^{n_\Delta} \rho_i(w\mu) \geq 0$ for $\mu \in \Sigma^+$, and hence, by (1), $\rho_i(w\mu) \geq 0$ for $\mu \in \Sigma^+$, $1 \leq i \leq n_\Delta$. So $\rho \cdot w$ is positive.

For uniqueness suppose that $\rho \in \text{Hom}(\Sigma, \Delta)$, $w \in W_\Sigma$ and both ρ and $\rho \cdot w$ are positive. Choose $\rho_1, \dots, \rho_{n_\Delta} \in \text{Hom}(Q_\Sigma, \mathbb{Z})$ as in the previous paragraph. Then $\rho_i(\mu) \geq 0$ and $(\rho_i \cdot w)(\mu) \geq 0$ for $\mu \in \Sigma^+$ and $1 \leq i \leq n_\Delta$. Thus ρ_i^* and $w^{-1} \rho_i^*$ are in $\overline{C(\Pi_\Sigma)}$. So, again since $\overline{C(\Pi_\Sigma)}$ is a fundamental domain for W_Σ , we have $\rho_i^* = w^{-1} \rho_i^*$ and hence $\rho_i = \rho_i \cdot w$. Thus $\rho = \rho \cdot w$. \square

Finally, it is clear that

$$\text{Aut}(\Pi_\Delta) \cdot \text{Hom}_p(\Sigma, \Delta) \cdot \text{Aut}(\Pi_\Sigma) \subseteq \text{Hom}_p(\Sigma, \Delta),$$

so $\text{Aut}(\Pi_\Sigma)$ (resp. $\text{Aut}(\Pi_\Delta)$) acts on the right (resp. left) on $\text{Hom}_p(\Sigma, \Delta)$ by composition.

1.4. The left action $*$ of $\text{Aut}(\Delta)$ on $\text{Hom}_p(\Sigma, \Delta)$. If $\theta \in \text{Aut}(\Delta)$ and $\rho \in \text{Hom}_p(\Sigma, \Delta)$, we let $\theta * \rho$ denote the unique element of $\theta \cdot \rho \cdot W_\Sigma$ that is contained in $\text{Hom}_p(\Sigma, \Delta)$ (see Lemma 1.3.1). That is

$$\{\theta * \rho\} = \text{Hom}_p(\Sigma, \Delta) \cap (\theta \cdot \rho \cdot W_\Sigma).$$

One checks easily that $*$: $(\theta, \rho) \mapsto \theta * \rho$ is a left action of $\text{Aut}(\Delta)$ on $\text{Hom}_p(\Sigma, \Delta)$, and that this action commutes with the right action \cdot of $\text{Aut}(\Pi_\Sigma)$ on the same set. (The latter fact uses only the observation that $\varphi^{-1} W_\Sigma \varphi \subseteq W_\Sigma$ for $\varphi \in \text{Aut}(\Pi_\Sigma)$.)

1.5. The sets $\text{Hom}_{\text{sh}}(\Sigma, \Delta)$ and $\text{Hom}_{\text{psh}}(\Sigma, \Delta)$. Suppose Δ is irreducible. Let

$$\text{Hom}_{\text{sh}}(\Sigma, \Delta) := \{\rho \in \text{Hom}(\Sigma, \Delta) : \rho(\Sigma) \cap \Delta_{\text{sh}} \neq \emptyset\}.$$

Note that $\text{Aut}(\Delta) \cdot \text{Hom}_{\text{sh}}(\Sigma, \Delta) \cdot \text{Aut}(\Sigma) \subseteq \text{Hom}_{\text{sh}}(\Sigma, \Delta)$, so $\text{Aut}(\Sigma)$ (resp. $\text{Aut}(\Delta)$) acts on the right (resp. left) on $\text{Hom}_{\text{sh}}(\Sigma, \Delta)$ by composition.

Also let

$$\text{Hom}_{\text{psh}}(\Sigma, \Delta) := \text{Hom}_p(\Sigma, \Delta) \cap \text{Hom}_{\text{sh}}(\Sigma, \Delta).$$

Again $\text{Aut}(\Pi_\Delta) \cdot \text{Hom}_{\text{psh}}(\Sigma, \Delta) \cdot \text{Aut}(\Pi_\Sigma) \subseteq \text{Hom}_{\text{psh}}(\Sigma, \Delta)$, and hence $\text{Aut}(\Pi_\Sigma)$ (resp. $\text{Aut}(\Pi_\Delta)$) acts on the right (resp. left) on $\text{Hom}_{\text{psh}}(\Sigma, \Delta)$ by composition.

Finally, it is clear that $\text{Aut}(\Delta) * \text{Hom}_{\text{psh}}(\Sigma, \Delta) \subseteq \text{Hom}_{\text{psh}}(\Sigma, \Delta)$, so we have a left action $*$ of $\text{Aut}(\Delta)$ on $\text{Hom}_{\text{psh}}(\Sigma, \Delta)$.

2. ROOT GRADED LIE ALGEBRAS

We will assume for the rest of the article that \mathbb{K} is a commutative associative ring of scalars containing $\frac{1}{6}$. (We will add the additional assumption that \mathbb{K} is an algebraically closed field of characteristic 0 in our classification results.)

All modules, algebras, trilinear pairs and triple systems will be assumed to be over \mathbb{K} .

2.1. Terminology for graded structures. For graded algebras and graded trilinear pairs, we will use the unmodified terms *simple* and *isomorphic* in the ungraded sense. More specifically, suppose G be an abelian group. A G -graded Lie algebra L will be said to be *simple* (resp. *graded simple*) if the only non-trivial proper ideals (resp. graded-ideals) of L are 0 and L . If L and L' are G -graded algebras, we will say that L is *isomorphic* (resp. *graded-isomorphic*) to L' , written $L \simeq L'$ (resp. $L \simeq_{\text{gr}} L'$), if there is an isomorphism (resp. graded-isomorphism) of L onto L' . We will use similar (and evident) terminology and notation for graded trilinear pairs (see Subsection 6.1).

If L is an algebra, two G -gradings of L are said to be *isomorphic* if the corresponding graded algebras are graded-isomorphic. (The modifier graded in not needed in this term since there is no ambiguity.) Again we will use similar terminology for gradings of trilinear pairs.

If L is a G -graded Lie algebra, an automorphism ω of L is said to be *grade reversing* if $\omega(L_\gamma) = L_{-\gamma}$ for $\gamma \in G$.

If L is a G -graded algebra and $\theta \in \text{Aut}(G)$, we let ${}^\theta L$ be the G -graded algebra such that ${}^\theta L = L$ as algebras and

$$({}^\theta L)_\alpha = L_{\theta^{-1}\alpha}$$

for $\alpha \in G$. We call ${}^\theta L$ the θ -image of L . Clearly ${}^1 L = L$ and

$$\theta_1({}^{\theta_2} L) = {}^{\theta_1 \theta_2} L \text{ for } \theta_1, \theta_2 \in \text{Aut}(G). \quad (2)$$

2.2. Root graded Lie algebras. Let Δ be a root system.

As in [AFS], a Δ -grading of a Lie algebra L will mean a Q_Δ -grading of L such that $\text{supp}_{Q_\Delta}(L) \subseteq \Delta \cup \{0\}$, where $\text{supp}_{Q_\Delta}(L)$ denotes the *support* of L in Q_Δ . In that case we call L together with the Δ -grading a Δ -graded Lie algebra. (We do not assume the existence of a grading subalgebra as in [ABG] and [BS], or equivalently a family of \mathfrak{sl}_2 -triples as in [N1].) Finally, if Δ is irreducible of type X_n , where $n = n_\Delta$, we sometimes call a Δ -graded Lie algebra an X_n -graded Lie algebra.

If Π_Δ is a base for Δ , we often use the basis Π_Δ for Q_Δ to identify Q_Δ with \mathbb{Z}^{n_Δ} , and in this way Δ -graded Lie algebras are \mathbb{Z}^{n_Δ} -graded Lie algebras.

If L is a Δ -graded Lie algebra and $\theta \in \text{Aut}(\Delta)$, then the θ -image ${}^\theta L$ of L is a Δ -graded Lie algebra. If $\theta \in W_\Delta$, we call ${}^\theta L$ a *Weyl image* of L .

3. ROOT GRADINGS OF FINITE DIMENSIONAL SEMI-SIMPLE LIE ALGEBRAS

Suppose in this section that \mathbb{K} is an algebraically closed field of characteristic 0, and that \mathcal{G} is a finite dimensional semisimple Lie algebra over \mathbb{K} .

Fix a Cartan subalgebra \mathcal{H} of \mathcal{G} , let $\mathcal{G} = \bigoplus_{\mu \in \mathcal{H}^*} \mathcal{G}_\mu$ be the root space decomposition of \mathcal{G} relative to \mathcal{H} , and let $\Sigma = \Sigma(\mathcal{G}, \mathcal{H})$ be the root system of \mathcal{G} relative to \mathcal{H} (see Subsection 1.1). We fix a base Π_Σ for this root system.

3.1. Gradings of \mathcal{G} by a finitely generated free abelian group. We first describe the G -gradings of \mathcal{G} up to isomorphism, where G is a free abelian group of finite rank written additively.

If $\rho \in \text{Hom}(Q_\Sigma, G)$, let $\mathcal{G}(\rho)$ be the G -graded Lie algebra whose underlying Lie algebra is \mathcal{G} and whose grading is defined by

$$\mathcal{G}(\rho)_\gamma = \sum_{\mu \in Q_\Sigma, \rho(\mu) = \gamma} \mathcal{G}_\mu \quad (3)$$

for $\gamma \in G$. We call this grading the ρ -grading of \mathcal{G} .

It is easily checked that

$${}^\theta \mathcal{G}(\rho) = \mathcal{G}(\theta \cdot \rho) \quad (4)$$

as G -graded Lie algebras for $\theta \in \text{Aut}(G)$ and $\rho \in \text{Hom}(Q_\Sigma, G)$.

Remark 3.1.1. The gradings on ${}^\theta L$ and $\mathcal{G}(\rho)$ defined above are each examples of *gradings induced by homomorphisms* as defined in [EK, §1.3]; and (2) and (4) follow from an evident general fact about induced gradings. Nevertheless, we use different notations for ${}^\theta L$ and $\mathcal{G}(\rho)$ to emphasize the different roles they play in our theory.

The first statement of the next proposition is a corollary of a more general result on graded algebras proved in [EK] using methods from algebraic groups (see [EK, Prop. 1.34 and Cor. 1.35]). It seems likely that the second and third statements are also known, although we are not aware of a reference. Nevertheless, for the reader's convenience, we give a proof of all three statements using Lie algebra methods.

Proposition 3.1.2. *Suppose that G is a finitely generated free abelian group.*

- (i) *Any G -grading of \mathcal{G} is isomorphic to a ρ -grading for some $\rho \in \text{Hom}(Q_\Sigma, G)$.*
- (ii) *If $\rho, \rho' \in \text{Hom}(Q_\Sigma, G)$, then the ρ -grading and the ρ' -grading are isomorphic if and only if ρ, ρ' lie in the same orbit in $\text{Hom}(Q_\Sigma, G)$ under the right action of $\text{Aut}(\Sigma)$ by composition.*
- (iii) *If $\mathcal{G} = \bigoplus_{\gamma \in G} \mathcal{G}_\gamma$ is a G -grading of \mathcal{G} , there exists a period 2 grade reversing automorphism of \mathcal{G} . So $\dim(\mathcal{G}_{-\gamma}) = \dim(\mathcal{G}_\gamma)$ for $\gamma \in G$.*

Proof. If $G = 0$ the statements are trivial, so we can assume that $G = \mathbb{Z}^\ell$, where $\ell \geq 1$.

(i) Suppose that \mathcal{G} is G -graded. Define derivations d_1, \dots, d_ℓ of \mathcal{G} by $d_i(x) = k_i x$ for $x \in \mathcal{G}_{(k_1, \dots, k_\ell)}$. Since \mathcal{G} is semisimple, every derivation is inner [H, Thm. 5.3], so we can write $d_i = \text{ad}(h_i)$ with $h_i \in \mathcal{G}$ for $1 \leq i \leq \ell$. Since the d_i are semisimple and commute, the h_i span an abelian ad-diagonalizable subalgebra of \mathcal{G} , and hence they are contained in a Cartan subalgebra \mathcal{H}' of \mathcal{G} [H, Cor. 15.3]. By [H, Cor. 16.4], we may choose $\eta \in \text{Aut}(\mathcal{G})$ with $\eta(\mathcal{H}') = \mathcal{H}$. Using η to transfer the grading, we can assume that $\mathcal{H}' = \mathcal{H}$. Define $\rho_i \in \text{Hom}(Q_\Sigma, \mathbb{Z})$ by $\rho_i(\mu) = \mu(h_i)$ for $\mu \in Q_\Sigma$. Then, $d_i(x) = [h_i, x] = \rho_i(\mu)x$ for $x \in \mathcal{G}_\mu$. Thus, the grading is the ρ -grading of \mathcal{G} , where $\rho(\alpha) = (\rho_1(\alpha), \dots, \rho_\ell(\alpha))$ for $\alpha \in Q_\Sigma$.

(ii) Suppose first that $\rho = \rho' \cdot \varphi$, where $\rho, \rho' \in \text{Hom}(Q_\Sigma, G)$, $\varphi \in \text{Aut}(\Sigma)$. Then [H, Thm. 14.2] tells us that there is $\eta \in \text{Aut}(\mathcal{G})$ with $\eta(\mathcal{G}_\mu) = \mathcal{G}_{\varphi(\mu)}$ for $\mu \in Q_\Sigma$. Hence for $\gamma \in G$ we have $\eta(\mathcal{G}(\rho)_\gamma) = \sum_{\mu \in Q_\Sigma, \rho(\mu)=\gamma} \mathcal{G}_{\varphi(\mu)} = \sum_{\nu \in Q_\Sigma, \rho'(\nu)=\gamma} \mathcal{G}_\nu = \mathcal{G}(\rho')_\gamma$. so the ρ and ρ' -gradings are isomorphic.

Conversely, suppose that $\rho, \rho' \in \text{Hom}(Q_\Sigma, G)$ and $\eta : \mathcal{G}(\rho) \rightarrow \mathcal{G}(\rho')$ is a G -graded-isomorphism. Clearly \mathcal{H} is a Cartan subalgebra of $\mathcal{G}(\rho)_0$, so $\eta(\mathcal{H})$ and \mathcal{H} are Cartan subalgebras of $\mathcal{G}(\rho')_0$. By Corollary 16.4 of [H], there exists $\eta' \in \text{Aut}(\mathcal{G})$ such that $\eta'\eta(\mathcal{H}) = \mathcal{H}$ and η' is a product of automorphisms of \mathcal{G} the form $\exp(\text{ad}(x))$, where $x \in \mathcal{G}(\rho')_0$ and the transformation $\text{ad}(x)$ of \mathcal{G} is nilpotent. Clearly, η' is a graded automorphism of $\mathcal{G}(\rho')$, so we can replace η by $\eta'\eta$ to assume that $\eta(\mathcal{H}) = \mathcal{H}$. Let \mathcal{H}^* be the dual space of \mathcal{H} and let $\eta^\sharp \in \text{GL}(\mathcal{H}^*)$ be the inverse dual of $\eta|_{\mathcal{H}} \in \text{GL}(\mathcal{H})$, so $(\eta^\sharp(\mu))(\eta(h)) = \mu(h)$ for $h \in \mathcal{H}$, $\mu \in \mathcal{H}^*$. Then $\eta(\mathcal{G}_\mu) = \mathcal{G}_{\eta^\sharp(\mu)}$ for $\mu \in \mathcal{H}^*$, so $\eta^\sharp(\Sigma) = \Sigma$. Hence $\varphi := \eta^\sharp|_{Q_\Sigma} \in \text{Aut}(\Sigma)$. Thus for $\mu \in \Sigma$, we have

$$\mathcal{G}_{\varphi(\mu)} = \eta(\mathcal{G}_\mu) \subseteq \eta(\mathcal{G}(\rho)_{\rho(\mu)}) = \mathcal{G}(\rho')_{\rho(\mu)}.$$

So $\rho'(\varphi(\mu)) = \rho(\mu)$ for $\mu \in \Sigma$, and hence $\rho = \rho' \cdot \varphi$.

(iii) By (i), we can assume that the given grading of \mathcal{G} is the ρ -grading where $\rho \in \text{Hom}(Q_\Sigma, G)$. Choose $\omega \in \text{Aut}(\mathcal{G})$ of period 2 of \mathcal{G} such that $\omega(\mathcal{G}_\mu) = \mathcal{G}_{-\mu}$ for $\mu \in Q_\Sigma$ [B, VIII, §4.4, Prop. 5]. Then, by (3), we have $\omega(\mathcal{G}(\rho)_\gamma) = \mathcal{G}(\rho)_{-\gamma}$ for $\gamma \in G$. \square

3.2. Classification of the root gradings of \mathcal{G} . The following lemma is clear from the definitions involved.

Lemma 3.2.1. *If Δ is a root system and $\rho \in \text{Hom}(Q_\Sigma, Q_\Delta)$, then the ρ -grading of \mathcal{G} is a Δ -grading if and only if $\rho \in \text{Hom}(\Sigma, \Delta)$. Moreover, if Δ is irreducible and $\rho \in \text{Hom}(\Sigma, \Delta)$, then $\mathcal{G}(\rho)_\alpha \neq 0$ for some $\alpha \in \Delta_{\text{sh}}$ if and only if $\rho \in \text{Hom}_{\text{sh}}(\Sigma, \Delta)$.*

We now classify the Δ -gradings of \mathcal{G} up to isomorphism in terms of orbits in $\text{Hom}_p(\Sigma, \Delta)$.

Theorem 3.2.2. *Suppose Δ is a root system with base Π_Δ . Then*

- (i) *Any Δ -grading of \mathcal{G} is isomorphic to the ρ -grading of \mathcal{G} for some $\rho \in \text{Hom}_p(\Sigma, \Delta)$, where $\text{Hom}_p(\Sigma, \Delta)$ is the space of positive homomorphisms of Σ into Δ relative to Π_Σ and Π_Δ .*
- (ii) *If ρ and τ are in $\text{Hom}_p(\Sigma, \Delta)$, then the ρ and τ -gradings of \mathcal{G} are isomorphic if and only if ρ and τ are in the same orbit in $\text{Hom}_p(\Sigma, \Delta)$ under the right action of $\text{Aut}(\Pi_\Sigma)$ by composition.*

Proof. (i): By Proposition 3.1.2(i), we can assume the grading is a ρ -grading for some $\rho \in \text{Hom}(Q_\Sigma, Q_\Delta)$. Then, by Lemma 3.2.1, $\rho \in \text{Hom}(\Sigma, \Delta)$. Thus, by Lemma 1.3.1 and Proposition 3.1.2(ii), we can assume that $\rho \in \text{Hom}_p(\Sigma, \Delta)$.

(ii): The implication “ \Leftarrow ” follows from Proposition 3.1.2(ii). For the converse, suppose that the ρ and τ -gradings are isomorphic. Then, by Proposition 3.1.2(ii), there exists $\varphi \in \text{Aut}(\Sigma)$ such that $\tau = \rho \cdot \varphi$. By [B, VI, § 1.5, Prop. 16], we may write $\varphi = \pi w$, where $\pi \in \text{Aut}(\Pi_\Sigma)$ and $w \in W_\Sigma$. So $\tau \cdot w^{-1} = \rho \cdot \pi$ is positive, and hence by uniqueness in Lemma 1.3.1, $\tau = \rho \cdot \pi$ as desired. \square

4. KANTOR PAIRS

We assume again that \mathbb{K} is a commutative associative ring containing $\frac{1}{6}$.

4.1. Trilinear pairs and triple systems. A *trilinear pair* is by definition a triple $(P, \{, , \}^-, \{, , \}^+)$, consisting of a pair $P = (P^-, P^+)$ of \mathbb{K} -modules together with two trilinear products $\{, , \}^\sigma : P^\sigma \times P^{-\sigma} \times P^\sigma \rightarrow P^\sigma$, $\sigma = \pm$. The map $\{, , \}^\sigma$ is called the σ -product for P , $\sigma = \pm$. We usually abbreviate $(P, \{, , \}^-, \{, , \}^+)$ as P or (P^-, P^+) .

If $P = (P^-, P^+)$ is a trilinear pair, the *opposite* of P is the trilinear pair P^{op} , with $(P^{\text{op}})^\sigma = P^{-\sigma}$ and with σ -product equal to the $-\sigma$ -product of P for $\sigma = \pm$.

A *triple system* (sometimes also called a *ternary algebra*) is a pair $(X, \{, , \})$ consisting of a \mathbb{K} -module X together with a trilinear product $\{, , \} : X \times X \times X \rightarrow X$. Again we usually write $(X, \{, , \})$ simply as X .

We have evident notions of ideal, simplicity and isomorphism for trilinear pairs [AFS, §2.1] and for triple systems.

There are two natural ways to obtain a trilinear pair from a triple system X . Indeed, if $\xi = \pm 1$, the ξ -double of a triple system X is the trilinear pair (X, X) with σ -products defined by $\{, , \}^+ = \{, , \}$ and $\{, , \}^- = \xi \{, , \}$. We call the 1-double (resp. the (-1) -double) of X the *double* of X (resp. the *signed double*) of X .

A *polarization* of a triple system X is a module decomposition $X = X^- \oplus X^+$ such that $\{X^\sigma, X^{-\sigma}, X^\sigma\} \subseteq X^\sigma$, $\{X^\sigma, X^\sigma, X\} = 0$ and $\{X, X^\sigma, X^\sigma\} = 0$ for $\sigma = \pm$. We say that X is *non-polarized* if X does not have a polarization.

As noted in [AFS, §2.1], the following fact is easy to verify:

Lemma 4.1.1. *Suppose X is a triple system and $\xi = \pm 1$. Then the ξ -double of X is a simple pair if and only if X is non-polarized and simple.*

We say that two triple systems X and X' are *isotopic* if there exist linear isomorphisms $\varphi^\sigma : X \rightarrow X'$ such that $\varphi^\sigma \{x, y, z\} = \{\varphi^\sigma x, \varphi^{-\sigma} y, \varphi^\sigma z\}$ for $\sigma = \pm$. (In [K1, K2], Kantor uses the term *weakly isomorphic* rather than isotopic.) Note that if X and X' are triple systems, then X and X' are isotopic if and only if their doubles (resp. their signed doubles) are isomorphic.

4.2. Kantor pairs. If P is a trilinear pair, $\sigma = \pm$, $x \in P^\sigma$, $y \in P^{-\sigma}$ and $z \in P^\sigma$, we define $D^\sigma(x, y) \in \text{End}(P^\sigma)$ and $K^\sigma(x, z) \in \text{Hom}(P^{-\sigma}, P^\sigma)$ by

$$D^\sigma(x, y)u = \{x, y, u\}^\sigma \quad \text{and} \quad K^\sigma(x, z)w = \{x, w, z\}^\sigma - \{z, w, x\}^\sigma.$$

for $u \in P^\sigma$, $w \in P^{-\sigma}$, $\sigma = \pm$. We then say that P is a *Kantor pair* [AF1, AFS] if the following identities hold:

$$\begin{aligned} [D^\sigma(x, y), D^\sigma(z, w)] &= D^\sigma(D^\sigma(x, y)z, w) - D^\sigma(z, D^{-\sigma}(y, x)w), \\ K^\sigma(x, z)D^{-\sigma}(w, u) + D^\sigma(u, w)K^\sigma(x, z) &= K^\sigma(K^\sigma(x, z)w, u) \end{aligned}$$

for $x, z, u \in P^\sigma$, $y, w \in P^{-\sigma}$, $\sigma = \pm$,

Clearly, the opposite of a Kantor pair is a Kantor pair.

A (linear) *Jordan pair* is a Kantor pair P such that $K^\sigma(P^\sigma, P^\sigma) = 0$ for $\sigma = \pm 1$. These pairs make up the best understood and most studied class of Kantor pairs [L, N1]. The reader can consult [AFS] (and its references) to see many other examples of Kantor pairs.

4.3. Freudenthal-Kantor triple systems. If $\xi = \pm 1$, a $(-\xi, 1)$ -*Freudenthal-Kantor triple system* (abbreviated as $(-\xi, 1)$ -FKTS), is a triple system X whose ξ -double is a Kantor pair [YO]. The defining identities for such a triple system are:

$$\begin{aligned} [L(x, y), L(z, w)] &= L(L(x, y)z, w) - \xi L(z, L(y, x)w), \\ \xi K(x, z)L(w, u) + L(u, w)K(x, z) &= K(K(x, z)w, u) \end{aligned}$$

for $x, y, z, w, u \in X$, where $L(x, y), K(x, z) \in \text{End}(X)$ are defined by $L(x, y)z = \{x, y, z\}$, $K(x, z)y = \{x, y, z\} - \{z, y, x\}$.

Example 4.3.1. We now mention two special cases which together with Remark 4.3.2 explain the names used in the term Freudenthal-Kantor triple system.

(i): We call a $(-1, 1)$ -FKTS a *Kantor triple system*, since these triple systems were studied in depth by Kantor in [K1, K2] (where they were called generalized Jordan triple systems of second order).

(ii): Suppose that \mathbb{K} is a field. A $(1, 1)$ -FKTS X is said to be *balanced* if $K(x, y) = \langle x, y \rangle \text{id}_X$ for some bilinear form $\langle \cdot, \cdot \rangle : X \times X \rightarrow \mathbb{K}$ [EKO, §1]. In that case $\langle \cdot, \cdot \rangle$ is a uniquely determined skew-symmetric form that we call the *skew form* of X . If $\langle \cdot, \cdot \rangle$ is non-degenerate, we see in the following remark that X is a Freudenthal triple system [M1] with a slightly modified product.

Remark 4.3.2. There are three other classes of triple systems which have been studied in the literature, whose definitions involve a skew form, and which are none other than balanced (1,1)-FKTS's with slightly modified products. To be more precise, suppose \mathbb{K} is a field, X is a finite dimensional vector space, $\langle , , \rangle : X \times X \times X \rightarrow X$ is a trilinear product and $\langle , \rangle : X \times X \rightarrow \mathbb{K}$ is a non-degenerate skew-symmetric form. Then $(X, \langle , , \rangle, \langle , \rangle)$ is a *balanced symplectic ternary algebra* [FF], a *symplectic triple system* [YA], or a *Freudenthal triple system* if and only if $(X, \{ , , \})$ is a balanced (1,1)-FKTS with skew form \langle , \rangle , where

$$\begin{aligned} \{x, y, z\} &= \langle z, x, y \rangle, \quad \{x, y, z\} = -\frac{1}{2}\langle x, y, z \rangle + \frac{1}{2}\langle x, y \rangle z \quad \text{or} \\ \{x, y, z\} &= -\frac{1}{2}\langle x, y, z \rangle + \frac{1}{2}\langle x, y \rangle z + \frac{1}{2}\langle z, x \rangle y + \frac{1}{2}\langle z, y \rangle x \end{aligned}$$

respectively. (This is easy to check in the first case. For the other two cases see [E1, Thm. 2.18] and [E2, Thm. 4.7].) Here for the definition of a Freudenthal triple system we use the one given in [M1] except that we allow the quartic form $\langle , \langle , , \rangle \rangle$ to be trivial; *in other words we allow the product $\langle , , \rangle$ in a Freudenthal triple system to be trivial.*

4.4. BC_1 -graded Lie algebras and Kantor pairs. The motivation for the study of Kantor pairs is their relationship with BC_1 -graded Lie algebras.

To recall this from [AFS] suppose that

$$\Delta = \Delta_{BC_1} := \{-2\alpha_1, -\alpha_1, \alpha_1, 2\alpha_1\}$$

is the irreducible root system of type BC_1 with base $\Pi_\Delta = \{\alpha_1\}$. We identify $Q_\Delta = \mathbb{Z}$ using the \mathbb{Z} -basis Π_Δ for Q_Δ . Then a BC_1 -graded Lie algebra is the same thing as a 5-graded Lie algebra $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$.

If $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ is a 5-graded Lie algebra, then $P = (L_{-1}, L_1)$ is a Kantor pair with products defined by

$$\{x, y, z\}^\sigma = [[x, y], z]$$

for $x, z \in L_{\sigma 1}$, $y \in L_{-\sigma 1}$, $\sigma = \pm$ [AF1, Thm. 7]. We call P the *Kantor pair enveloped by the 5-graded Lie algebra L* , and we say that *the 5-graded Lie algebra L envelops P* .

Note that $\Delta_{sh} = \{\pm\alpha_1\}$. So if L is a 5-graded Lie algebra and P is the Kantor pair enveloped by L , then $P^- \oplus P^+ = \bigoplus_{\alpha \in \Delta_{sh}} L_\alpha$ in L .

Every Kantor pair is enveloped by some 5-graded Lie algebra. Indeed, given a Kantor pair P , there exists a 5-graded Lie algebra $\mathfrak{K}(P)$, which is unique up to graded-isomorphism, such that $\mathfrak{K}(P)$ envelops P , $\mathfrak{K}(P)$ is generated as an algebra by T , where $T = \mathfrak{K}(P)_{-1} + \mathfrak{K}(P)_1$, and the centre of $\mathfrak{K}(P)$ intersects trivially with $[T, T]$ (see [AFS, Cor. 3.5.2]). The 5-graded Lie algebra $\mathfrak{K}(P)$ is called the *Kantor Lie algebra of P* , and it is constructed explicitly in [AF1, §3–4] (see also [AFS, §3.3]) with

$$\mathfrak{K}(P)_{\sigma 1} \simeq_{\mathbb{K}\text{-mod}} P^\sigma, \quad \mathfrak{K}(P)_{\sigma 2} \simeq_{\mathbb{K}\text{-mod}} K^\sigma(P^\sigma, P^\sigma), \quad \mathfrak{K}(P)_0 = [\mathfrak{K}(P)_{-1}, \mathfrak{K}(P)_1] \quad (5)$$

for $\sigma = \pm$, where $\simeq_{\mathbb{K}\text{-mod}}$ indicates isomorphism as \mathbb{K} -modules.

Note that isomorphisms of Kantor pairs induce graded isomorphisms of their Kantor Lie algebras (see [AFS, §3.5]), and hence two Kantor pairs are isomorphic if and only if their Kantor Lie algebras are graded-isomorphic.

Lemma 4.4.1. *Suppose that P is a Kantor pair and $\xi = \pm 1$. If P is isomorphic to the ξ -double of a trilinear pair, then there is grade reversing automorphism ω of $\mathfrak{K}(P)$ such that $\omega^2|_{P^-+P^+} = \xi \text{id}_{P^-+P^+}$. Conversely if L is a 5-graded Lie algebra that envelops P and there is grade reversing automorphism ω of L satisfying $\omega^2|_{P^-+P^+} = \xi \text{id}_{P^-+P^+}$, then P is isomorphic to the ξ -double of the triple system P^- with product $\{x, y, z\} = \{x, \omega y, z\}^-$.*

Proof. (See also [L, pp. 5–6] and [At, Thm. 1] for related observations.) For the first statement we can assume that P equals the ξ -double of a trilinear pair X . Then $(\text{id}_X, \xi \text{id}_X)$ is an isomorphism of P onto P^{op} , which therefore induces an isomorphism ω of $\mathfrak{K}(P)$ onto $\mathfrak{K}(P^{\text{op}})$. Since $\mathfrak{K}(P^{\text{op}})$ is $\mathfrak{K}(P)$ with the grading reversed, we have the first statement. For the second statement, an easy calculation shows that $(\text{id}_{P^-}, \omega|_{P^+})$ is an isomorphism of P onto the ξ -double of the triple system P^- with the indicated product. \square

If P is a Kantor pair, then [GLN, Prop. 2.7(iii)]

$$P \text{ is simple if and only if } \mathfrak{K}(P) \text{ is simple.} \quad (6)$$

Hence, any simple Kantor pair is enveloped by a simple 5-graded Lie algebra. Conversely we have the following proposition:

Proposition 4.4.2. [AFS, Thm. 3.5.5 and Lemma 3.2.3] *Let P be a nonzero Kantor pair and suppose that L is a simple 5-graded Lie algebra that envelops P . Then P is simple, $L \simeq_{gr} \mathfrak{K}(P)$ and*

$$L = [L_{-1}, L_{-1}] \oplus L_{-1} \oplus [L_{-1}, L_1] \oplus L_1 \oplus [L_1, L_1] \quad (7)$$

4.5. Balanced dimension and close-to-Jordan pairs. Suppose in this subsection that \mathbb{K} is a field.

We say that a Kantor pair P is *finite dimensional* if each P^σ is finite dimensional. In view of (5), P is finite dimensional if and only if $\mathfrak{K}(P)$ is finite dimensional.

As in [AFS, §2.1], we say that a Kantor pair P has *balanced dimension*, if $\dim(P^+) = \dim(P^-) = d$, where d is a non-negative integer, in which case we call d the *balanced dimension* of P . Similarly, as in [AFS, §3.8], we say P has *balanced 2-dimension*, if $\dim(K^+(P^+, P^+)) = \dim(K^-(P^-, P^-)) = e$, where e is a non-negative integer, in which case we call e the *balanced 2-dimension* of P . Note that by (5), the balanced dimension of P (resp. the balanced 2-dimension of P), is given, when it is defined, by $d = \dim(\mathfrak{K}(P)_{\sigma_1})$ (resp. $e = \dim(\mathfrak{K}(P)_{\sigma_2})$) for $\sigma = \pm$.

Lemma 4.5.1. *If P is finite dimensional and $P^{\text{op}} \simeq P$, then P has balanced dimension and balanced 2-dimension.*

Proof. We have $\dim(P^+) = \dim((P^{\text{op}})^-) = \dim(P^-)$ and $\dim(K^+(P^+, P^+)) = \dim(K^-(P^-, P^-)) = \dim(K^-(P^{\text{op}})^-, (P^{\text{op}})^-)) = \dim(K^-(P^-, P^-))$. \square

Note that a Kantor pair P is Jordan if and only if P has balanced 2-dimension 0. With this in mind, we view balanced 2-dimension (when it is defined) as a measure of distance from Jordan theory. In this spirit, we make the following definition.

Definition 4.5.2. A *close-to-Jordan Kantor pair*, or simply a *close-to-Jordan pair*, is a Kantor pair of balanced 2-dimension 1.

4.6. Finite dimensional simple Kantor pairs. Suppose in this subsection that \mathbb{K} is an algebraically closed field of characteristic 0.

Proposition 4.6.1. *A trilinear pair P is a finite dimensional simple Kantor pair if and only if it is isomorphic to the double of a finite dimensional non-polarized simple Kantor triple system. Moreover in that case $P^{\text{op}} \simeq P$, and hence P has balanced dimension and balanced 2-dimension.*

Proof. By Lemma 4.1.1, we have the implication “ \Leftarrow ” in the first statement. Suppose conversely that P is a finite dimensional simple Kantor pair. By (6) and Proposition 3.1.2(iii), we can choose a period 2 grade reversing $\omega \in \text{Aut}(\mathfrak{K}(P))$. Then ω exchanges P^+ and P^- , so $(\omega|_{P^-}, \omega|_{P^+})$ is an isomorphism of P onto P^{op} . The rest now follows from Lemmas 4.4.1 and 4.5.1 \square

Remark 4.6.2. We see from Proposition 4.6.1 that the problem of classifying finite dimensional simple Kantor pairs up to isomorphism is equivalent to the problem of classifying finite dimensional non-polarized simple Kantor triple systems up to isotopy.

If P is a finite dimensional simple Kantor pair over \mathbb{K} , we define the *type* of P to be the type of the simple Lie algebra $\mathfrak{K}(P)$.

Proposition 4.6.3. *Suppose P is a nonzero Kantor pair. If L is any finite dimensional simple 5-graded Lie algebra that envelops P , then P is finite dimensional and simple, $L \simeq_{gr} \mathfrak{K}(P)$, the type of P is the same as the type of L , the balanced dimension of P equals $\dim(L_{\sigma 1})$ for $\sigma = \pm$, and the balanced 2-dimension of P equals $\dim(L_{\sigma 2})$ for $\sigma = \pm$.*

Proof. This follows from Propositions 4.4.2 and 4.6.1. \square

5. CLASSIFICATION OF FINITE DIMENSIONAL SIMPLE KANTOR PAIRS

Suppose in this section that \mathbb{K} is an algebraically closed field of characteristic 0 and that Π is the connected Dynkin diagram of type X_n . We will use certain admissible subsets of Π , or equivalently certain markings of Π , to classify the finite dimensional simple Kantor pairs over \mathbb{K} .

For this purpose, we let \mathcal{G} be a finite dimensional simple Lie algebra over \mathbb{K} of type X_n with Cartan subalgebra \mathcal{H} and root system $\Sigma = \Sigma(\mathcal{G}, \mathcal{H})$ relative to \mathcal{H} (see Subsection 1.1); and let Π_Σ be a base for this root system. Then there exists a diagram isomorphism $\iota : \Pi \rightarrow \Pi_\Sigma$. For simplicity we use ι to identify $\Pi = \Pi_\Sigma$.

Let μ^+ be the highest root of Σ relative to Π , $\mu^- := -\mu^+$, and

$$\tilde{\Pi} := \Pi \cup \{\mu^-\}.$$

The Dynkin diagram for $\tilde{\Pi}$ (or simply $\tilde{\Pi}$) is called the *extended Dynkin diagram* (or completed Dynkin graph in [B]) for Π .

We also assume as in Subsection 4.4 that

$$\Delta = \Delta_{\text{BC}_1} := \{-2\alpha_1, -\alpha_1, \alpha_1, 2\alpha_1\}$$

is the irreducible root system of type BC_1 with base $\Pi_\Delta = \{\alpha_1\}$, and we identify $Q_\Delta = \mathbb{Z}$ using the \mathbb{Z} -basis Π_Δ for Q_Δ . Note that $\text{Aut}(\Delta) = W_\Delta = \{\pm 1\}$.

5.1. Kantor-admissible subsets of Π . To emphasize the role of BC_1 here, we write the sets $\text{Hom}(\Sigma, \Delta)$, $\text{Hom}_{\text{sh}}(\Sigma, \Delta)$, $\text{Hom}_{\text{p}}(\Sigma, \Delta)$, and $\text{Hom}_{\text{psh}}(\Sigma, \Delta)$ respectively as $\text{Hom}(\Sigma, BC_1)$, $\text{Hom}_{\text{sh}}(\Sigma, BC_1)$, $\text{Hom}_{\text{p}}(\Sigma, BC_1)$ and $\text{Hom}_{\text{psh}}(\Sigma, BC_1)$. Then we have

$$\begin{aligned} \text{Hom}(\Sigma, BC_1) &= \{\rho \in \text{Hom}(Q_\Sigma, \mathbb{Z}) : \rho(\Sigma) \subseteq \{0, \pm 1, \pm 2\}\}, \\ \text{Hom}_{\text{sh}}(\Sigma, BC_1) &= \{\rho \in \text{Hom}(\Sigma, BC_1) : 1 \in \rho(\Sigma)\}. \end{aligned} \quad (8)$$

If $S \subseteq \Pi$, we define $\chi_S \in \text{Hom}(Q_\Sigma, \mathbb{Z})$ by

$$\chi_S(\lambda) = \begin{cases} 1 & \text{if } \lambda \in S \\ 0 & \text{if } \lambda \in \Pi \setminus S, \end{cases} \quad (9)$$

and we set $\chi_\lambda = \chi_{\{\lambda\}}$ for $\lambda \in \Pi$. Note that $\chi_\emptyset = 0$.

A subset S of Π is said to be *Kantor-admissible* if $\chi_S \in \text{Hom}_{\text{sh}}(\Sigma, BC_1)$ (or equivalently $\chi_S \in \text{Hom}_{\text{psh}}(\Sigma, BC_1)$). Let

$\text{KA}(\Pi) =$ the set of all Kantor-admissible subsets of Π .

Remark 5.1.1. The set $\text{KA}(\Pi)$ (or equivalently the term Kantor-admissible) has been defined using our choice of \mathcal{G} , \mathcal{H} , Π_Σ and $\iota : \Pi \rightarrow \Pi_\Sigma$. However, it is easy to see (since Dynkin diagram isomorphisms induce root system isomorphisms), that a different choice of these objects leads to the same set $\text{KA}(\Pi)$. In short, $\text{KA}(\Pi)$ is *well-defined*.

Proposition 5.1.2. *The map $S \mapsto \chi_S$ is a bijection of the set $\text{KA}(\Pi)$ onto the set $\text{Hom}_{\text{psh}}(\Sigma, BC_1)$.*

Proof. All but surjectivity is clear. For surjectivity, suppose $\rho \in \text{Hom}_{\text{psh}}(\Sigma, BC_1)$. Then $1 \in \rho(\Sigma^+)$, so $1 \in \rho(\Pi)$. Note also that $\sum_{\lambda \in \Pi} \lambda \in \Sigma^+$, since Σ is irreducible. So $0 \leq \sum_{\lambda \in \Pi} \rho(\lambda) \leq 2$. At least one term in this sum is 1, so all are 0 or 1. Let $S = \{\lambda \in \Pi : \rho(\lambda) = 1\}$. Then ρ and χ_S agree on Π , so $\rho = \chi_S$. Finally $S \in \text{KA}(\Pi)$ by definition. \square

We have the following characterization of Kantor-admissible subsets of Π .

Proposition 5.1.3. *If S is a subset of Π , then the following are equivalent:*

- (a) $S \in \text{KA}(\Pi)$.
- (b) $S \neq \emptyset$ and $\chi_S \in \text{Hom}(\Sigma, BC_1)$.
- (c) $\chi_S(\mu^+) \in \{1, 2\}$.
- (d) $S = \{\lambda\}$ with $\chi_\lambda(\mu^+) \in \{1, 2\}$; or $S = \{\lambda, \lambda'\}$ with $\lambda \neq \lambda'$ and $\chi_\lambda(\mu^+) = \chi_{\lambda'}(\mu^+) = 1$.

Proof. The implications in cyclic order are clear using (8). \square

We define a right action of $\text{Aut}(\Pi)$ on the set of subsets of Π by

$$S \cdot \varphi := \varphi^{-1}(S).$$

One checks that

$$\chi_{S \cdot \varphi} = \chi_S \cdot \varphi. \quad (10)$$

for $S \subseteq \Pi$ and $\varphi \in \text{Aut}(\Pi)$. Hence the right action of $\text{Aut}(\Pi)$ stabilizes the set $\text{KA}(\Pi)$, so we have a right action \cdot of $\text{Aut}(\Pi)$ on $\text{KA}(\Pi)$.

5.2. Classification. We next use Kantor's approach for constructing Kantor triple systems [K1] to construct Kantor pairs.

Construction 5.2.1. (The Kantor pair $\mathfrak{P}(\Pi; S)$) Let $S \in \text{KA}(\Pi)$. Then $\chi_S \in \text{Hom}_{\text{sh}}(\Sigma, \text{BC}_1)$, so $\mathcal{G}(\chi_S)$ is a 5-graded Lie algebra with $\mathcal{G}(\chi_S)_{-1} \neq 0$ or $\mathcal{G}(\chi_S)_1 \neq 0$ by Lemma 3.2.1. Let $\mathfrak{P}(\Pi; S)$ be the Kantor pair enveloped by $\mathcal{G}(\chi_S)$ (see Section 4.4). Then $\mathfrak{P}(\Pi; S) = (\mathfrak{P}(\Pi; S)^-, \mathfrak{P}(\Pi; S)^+)$ is nonzero and given explicitly by

$$\mathfrak{P}(\Pi; S)^\sigma = \mathcal{G}(\chi_S)_{\sigma 1} = \sum_{\mu \in \Sigma, \chi_S(\mu) = \sigma 1} \mathcal{G}_\mu$$

for $\sigma = \pm$ with products given by $\{x, y, z\}^\sigma = [[x, y], z]$ in \mathcal{G} .

Remark 5.2.2. The definition of $\mathfrak{P}(\Pi; S)$ uses our choice of \mathcal{G} , \mathcal{H} , Π_Σ and $\iota : \Pi \rightarrow \Pi_\Sigma$. It is easy to see (since Dynkin diagram isomorphisms induce Lie algebra isomorphisms) that a different choice of these objects leads to a Kantor pair that is isomorphic to $\mathfrak{P}(\Pi; S)$. That is to say, $\mathfrak{P}(\Pi; S)$ is well-defined up to isomorphism.

Since Π is fixed in our discussion, we will usually abbreviate our notation and write $\mathfrak{P}(\Pi; S)$ simply as $\mathfrak{P}(S)$.

Remark 5.2.3. Let $S \in \text{KA}(\Pi)$.

(i) By Proposition 4.6.3, $\mathfrak{P}(S)$ is a finite dimensional simple Kantor pair of type X_n and $\mathcal{G}(\chi_S) \simeq_{\text{gr}} \mathfrak{K}(\mathfrak{P}(S))$.

(ii) Again using Proposition 4.6.3, we see that the balanced dimension and the balanced 2-dimension of $\mathfrak{P}(S)$ equal respectively $\dim(\mathcal{G}(\chi_S)_1)$ and $\dim(\mathcal{G}(\chi_S)_2)$. Hence these quantities are given respectively by

$$|\{\mu \in \Sigma : \chi_S(\mu) = 1\}| \quad \text{and} \quad |\{\mu \in \Sigma : \chi_S(\mu) = 2\}|.$$

(iii) $\mathfrak{P}(S)$ is Jordan if and only if it has balanced 2-dimension 0, which by (ii) holds if and only if $\chi_S(\mu^+) = 1$.

We now state a classification theorem for finite dimensional simple Kantor pairs in terms of Kantor admissible subsets of Π .

Theorem 5.2.4 (Kantor). *Suppose Π is the connected Dynkin diagram of type X_n .*

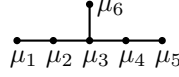
- (i) *If $S \in \text{KA}(\Pi)$, then $\mathfrak{P}(S)$ is a finite dimensional simple Kantor pair of type X_n .*
- (ii) *If P is a finite dimensional simple Kantor pair of type X_n , then P is isomorphic to $\mathfrak{P}(S)$ for some $S \in \text{KA}(\Pi)$.*
- (iii) *If S and S' are in $\text{KA}(\Pi)$, then $\mathfrak{P}(S)$ and $\mathfrak{P}(S')$ are isomorphic if and only if S and S' are in the same orbit in $\text{KA}(\Pi)$ under the right action of $\text{Aut}(\Pi)$.*

We have seen (i) above; and (ii) and (iii) follow easily using Theorem 3.2.2, Proposition 4.6.3 and Proposition 5.1.2. We omit the details since we will obtain the theorem as a special case of our classification of finite dimensional simple SP-graded Kantor pairs (see Theorem 7.2.4 and Remark 7.2.5).

Remark 5.2.5. We have attributed Theorem 5.2.4 to Isai Kantor, since in [K1, §4] he proved the equivalent classification result (see Remark 4.6.2) for finite dimensional non-polarized simple Kantor triple systems up to isotopy, although he omitted some details both in his statements and proofs. We note that Kantor also gave models of each of the Kantor triple systems that he considered ([K1, §5–6], [K2]). We will not look at these models in general, but rather focus on models for the reflections of simple close-to-Jordan pairs later in Section 9.

5.3. Marked Dynkin diagrams for simple Kantor pairs. We represent $S \in \text{KA}(\Pi)$ by the Dynkin diagram for Π with the nodes in S marked with a circle; and we use the same marked Dynkin diagram to represent the Kantor pair $\mathfrak{P}(S)$.

Example 5.3.1. (Type E_6). Suppose that $\Pi = \{\mu_1, \dots, \mu_6\}$ is of type E_6 with Dynkin diagram



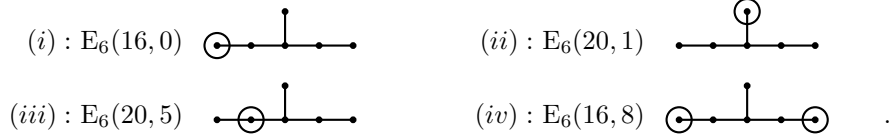
Now $\mu^+ = \mu_1 + 2\mu_2 + 3\mu_3 + 2\mu_4 + \mu_5 + 2\mu_6$, and the extended Dynkin diagram $\tilde{\Pi}$ is



[B, VI, §4.12, (IV)]. So by Proposition 5.1.3(d) there are, up to the right action of $\text{Aut}(\Pi)$, four Kantor-admissible subsets S of Π :

$$(i) : \{\mu_1\}, \quad (ii) : \{\mu_6\}, \quad (iii) : \{\mu_2\}, \quad (iv) : \{\mu_1, \mu_5\}.$$

These are represented respectively by the following marked diagrams



So by Theorem 5.2.4 there are four simple Kantor pairs of type E_6 up to isomorphism, which are represented by these marked diagrams.

For each S above, we have labelled the marked diagram representing S and $\mathfrak{P}(S)$ by $E_6(d, e)$, where d is the balanced dimension of $\mathfrak{P}(S)$ and e is the balanced 2-dimension of $\mathfrak{P}(S)$. (d and e were computed using Remark 5.2.3(ii) and a list of roots in Σ [B, Plate V].) We will sometimes also use the label $E_6(d, e)$ for the Kantor pair $\mathfrak{P}(S)$ itself.

Note that the unique pair in the list that is Jordan is $E_6(16, 0)$, and the unique pair in the list that is close-to-Jordan is $E_6(20, 1)$.

For each of the other types X_n , it is easy using the same method to write down the marked Dynkin diagrams representing the simple Kantor pairs of type X_n up to isomorphism.

5.4. The close-to-Jordan case. We next consider the classification in the special case of simple close-to-Jordan pairs of type X_n . By Theorem 5.2.4, there is only one simple finite dimensional Kantor pair of type A_1 , and it is Jordan so not close-to-Jordan. For the other types, we have the following:

Theorem 5.4.1 (Kantor and Skopec). *Suppose that $X_n \neq A_1$, and let S be the set of nodes of Π that are adjacent to μ^- in $\tilde{\Pi}$.*

- (i) $\chi_S(\mu) = \langle \mu, \mu^+ \rangle$ for $\mu \in Q_\Sigma$; so $\chi_S(\mu^+) = 2$.
- (ii) S is the unique element of $\text{KA}(\Pi)$ such that $\mathfrak{P}(S)$ is close-to-Jordan.
- (iii) $\mathfrak{P}(S)$ is, up to isomorphism, the unique finite dimensional simple close-to-Jordan pair of type X_n .

Proof. Although this result is not stated in this form in [KS], it is implicit in the statement and proof of Theorem 4 in [KS]. For the reader's convenience we give a different proof.

(i): We can assume that $\mu \in \Pi$. So $\mu \notin \mathbb{Q}\mu^+$ (since $X_n \neq A_1$). Hence $\langle \mu, \mu^+ \rangle \in \{0, 1\}$ by [B, VI, §1.8, Prop. 25(iv)]; whereas $\chi_S(\mu) \in \{0, 1\}$ by definition of χ_S . Finally $\langle \mu, \mu^+ \rangle \neq 0$ iff $\mu \in S$ iff $\chi_S(\mu) \neq 0$.

(ii): If $S' \in \text{KA}(\Pi)$, recall that by Remark 5.2.3(ii), $\mathfrak{P}(S')$ is close-to-Jordan if and only if $\{\mu \in \Sigma : \chi_{S'}(\mu) = 2\} = \{\mu^+\}$.

Now $S \in \text{KA}(\Pi)$ by Proposition 5.1.3(c). To see that $\mathfrak{P}(S)$ is close-to-Jordan, it is enough to show that $\chi_S(\mu) \neq 2$ for $\mu \in \Sigma^+ \setminus \{\mu^+\}$. But if $\mu \in \Sigma^+ \setminus \{\mu^+\}$, there exists $\kappa \in \Pi$ such that $\mu \in \mu^+ - \kappa - \sum_{\lambda \in \Pi} \mathbb{Z}_{\geq 0} \lambda$ and $\mu^+ - \kappa \in \Sigma$. Thus, $\kappa \in S$, so $\chi_S(\mu) \leq 2 - 1 = 1$.

Finally, suppose that $S' \in \text{KA}(\Pi)$ and $\mathfrak{P}(S')$ is close-to-Jordan. So $\chi_{S'}(\mu^+) = 2$. If there exists $\kappa \in S$ such that $\kappa \notin S'$, then $\mu^+ - \kappa \in \Sigma^+ \setminus \{\mu^+\}$ and $\chi_{S'}(\mu^+ - \kappa) = 2$, a contraction. So $S \subseteq S'$. Moreover, this inclusion is not proper, since otherwise we would have $2 = \chi_S(\mu^+) < \chi_{S'}(\mu^+)$. So $S' = S$.

(iii) follows from (ii) and Theorem 5.2.4. \square

Example 5.3.1 provides an illustration of Theorem 5.4.1(iii) in type E_6 .

Remark 5.4.2. Suppose we have the assumptions of Theorem 5.4.1.

(i) One sees checking types case-by-case that $\text{card}(S) = 1$ if $X_n \neq A_n$.

(ii) The 5-grading of $\mathcal{G}(\chi_S)$ induces a 5-grading of the set $\Sigma \cup \{0\}$ (which is viewed as a root system in [LN]). Since $\chi_S(\lambda) = \langle \lambda, \mu^+ \rangle$ for $\lambda \in Q_\Sigma$, one sees that this 5-grading of $\Sigma \cup \{0\}$ is the one described previously in [LN, §17.10]. We won't need this fact, so we omit the details.

5.5. Close-to-Jordan pairs and Freudenthal-Kantor triple systems. Suppose in this subsection that $S \in \text{KA}(\Pi)$ and $P = \mathfrak{P}(S)$ is close-to-Jordan. Recall by Theorem 5.4.1(ii) that $X_n \neq A_1$ and S is the set of nodes of Π that are adjacent to μ^- in $\tilde{\Pi}$.

Choose nonzero $e^\sigma \in \mathcal{G}_{\mu^\sigma}$ for $\sigma = \pm$ such that

$$[h^+, e^\sigma] = \sigma 2e^\sigma, \text{ where } h^+ = [e^+, e^-].$$

Then $\mathcal{S} := \mathbb{K}e^- \oplus \mathbb{K}h^+ \oplus e^+ \simeq \mathfrak{sl}_2(\mathbb{K})$. Also, since P is close-to-Jordan, we have

$$\mathcal{G}(\chi_S)_{\sigma 2} = \mathbb{K}e^\sigma \quad (12)$$

for $\sigma = \pm$. We set

$$\omega := \exp(\text{ad } e^+) \exp(-\text{ad } e^-) \exp(\text{ad } e^+) \in \text{Aut}(\mathcal{G}),$$

in which case

$$\omega(e^\sigma) = -e^{-\sigma}, \sigma = \pm, \text{ and } \omega(h^+) = -h^+. \quad (13)$$

We now use \mathfrak{sl}_2 -theory to verify some properties of the triple (e^-, h^+, e^+) and the automorphism ω .

Lemma 5.5.1. *We have*

- (i) $\mu(h^+) = \langle \mu, \mu^+ \rangle = \chi_S(\mu)$ for $\mu \in Q_\Sigma$.
- (ii) $\mathcal{G}(\chi_S)_i = \{g \in \mathcal{G} : [h^+, g] = ig\}$ for $i \in \mathbb{Z}$.
- (iii) ω reverses the 5-grading of $\mathcal{G}(\chi_S)$, and hence $\omega(P^\sigma) = P^{-\sigma}$ for $\sigma = \pm$.
- (iv) $\omega^2|_{P^- + P^+} = -\text{id}_{P^- + P^+}$.
- (v) $\omega(x) = -\sigma[e^{-\sigma}, x]$ for $x \in P^\sigma$, $\sigma = \pm$.

(vi) If $\sigma = \pm$, there exists a unique bilinear form $\zeta^\sigma : P^\sigma \times P^{-\sigma} \rightarrow \mathbb{K}$ such that

$$[[x, a], e^\sigma] = \zeta^\sigma(x, a)e^\sigma \quad (14)$$

for $x \in P^\sigma$, $a \in P^{-\sigma}$. Further

$$[x, \omega a] = -\sigma \zeta^\sigma(x, a)e^\sigma, \quad (15)$$

$$\zeta^\sigma(x, a) = \zeta^{-\sigma}(a, x) \quad (16)$$

for $x \in P^\sigma$, $a \in P^{-\sigma}$; and ζ^σ is nondegenerate.

(vii) If $x, y \in P^\sigma$ and $a \in P^{-\sigma}$, then

$$[[\omega x, a], y] = \zeta^\sigma(x, a)\omega y \quad \text{and} \quad \omega([x, a], y) = \zeta^\sigma(x, a)\omega y + [[x, a], \omega y].$$

(viii) If $x, y, z \in P^\sigma$, then $[[x, \omega y], z] - [[z, \omega y], x] = -\zeta^\sigma(x, \omega z)y$.

Proof. (i): The first equality is standard [H, Prop. 8.2(g)] and the second was seen in Theorem 5.4.1(i).

(ii) and (iii): (ii) follows from (i), and (iii) follows from (ii) and (13).

(iv) and (v): Let $T = P^- \oplus P^+$ in \mathcal{G} . Then T is an \mathcal{S} -submodule of \mathcal{G} under the adjoint action. Also, by (ii), P^- and P^+ are respectively the -1 and 1 eigenspaces for $\text{ad}(h^+)$ in T . So, by \mathfrak{sl}_2 -theory [B, VIII, §1.2–1.3], the \mathcal{S} -module T is the direct sum of copies of the 2-dimensional irreducible \mathcal{S} -module. Then (iv) and (v) follow by well known 2×2 -matrix calculations (see for example [B, VIII, §1.5] with $X_\sigma = \sigma e^\sigma$).

(vi): The first statement follows from (12). Then $[x, \omega a] = \sigma[x, [e^\sigma, a]] = -\sigma \zeta^\sigma(x, a)e^\sigma$; and (16) follows by applying ω and using (13) and (iv). For nondegeneracy, it is enough to show that if $x \in P^\sigma$ and $[x, P^\sigma] = 0$, then $x = 0$. For this, let κ be the Killing form. Then $0 = \kappa([x, P^\sigma], e^{-\sigma}) = \kappa(x, [P^\sigma, e^{-\sigma}]) = \kappa(x, P^{-\sigma})$ by (iii) and (v). But $\mathcal{G}(\chi_S)_1$ and $\mathcal{G}(\chi_S)_{-1}$ are paired by κ , so $x = 0$.

(vii): Using (15), (16) and (v), we have $[[\omega x, a], y] = -\sigma \zeta^{-\sigma}(a, x)[e^{-\sigma}, y] = \zeta^\sigma(x, a)\omega y$. Then, since $\text{ad}(e^{-\sigma})$ is a derivation of \mathcal{G} which kills $P^{-\sigma}$, we have $\omega([x, a], y) = [[\omega x, a], y] + [[x, a], \omega y] = \zeta^\sigma(x, a)\omega y + [[x, a], \omega y]$.

(viii): Using (iv), the first identity of (vii) and (16), we have $[[x, \omega y], z] - [[z, \omega y], x] = [[x, z], \omega y] = [[\omega^2 z, x], \omega y] = \zeta^{-\sigma}(\omega z, x)\omega^2 y = -\zeta^\sigma(x, \omega z)y$. \square

Theorem 5.5.2. *A trilinear pair is a finite dimensional simple close-to-Jordan pair if and only if it is isomorphic to the signed double of a nonzero finite dimensional balanced $(1, 1)$ -FKTS with non-degenerate skew-form (see Example 4.3.1(ii)).*

Proof. “ \Rightarrow ” By Theorem 5.4.1(iii) we can assume that the trilinear pair is $P = \mathfrak{P}(S)$. Then by Lemma 4.4.1 and Lemma 5.5.1 (parts (iii) and (iv)), P is isomorphic to the signed double of the triple system $X = P^-$ with product $\{a, b, c\} := \{a, \omega b, c\}^-$. So by definition X is a $(1, 1)$ -FKTS. Moreover X is balanced with skew form $\langle a, b \rangle = -\zeta^-(a, \omega b)$ by Lemma 5.5.1(viii); \langle, \rangle is nondegenerate by Lemma 5.5.1(vi); and $X \neq 0$ since P is simple.

“ \Leftarrow ” Conversely suppose that X is a non-zero balanced $(1, 1)$ -FKTS with non-degenerate skew form. Then, we can use the construction from [F, Thm. 1] of a simple 5-graded Lie algebra $\mathfrak{S} = \mathfrak{S}(X, \mathfrak{R}(X))$ with $\dim(\mathfrak{S})_{\sigma_2} = 1$. (Actually, the construction starts with a balanced symplectic ternary algebra, but we have seen the translation in Remark 4.3.2.) One can easily check that \mathfrak{S} envelops the signed-double of X . Hence the signed-double of X is a simple close-to-Jordan pair by Proposition 4.6.3. \square

In [FF, Cor. 2], the authors used the work of Meyberg [M1] on Freudenthal triple systems, to give constructions (mainly as 2×2 matrix systems with Jordan entries) of all balanced symplectic algebras with nondegenerate skew forms. (Note however that there is a missing term $(a_1 \times a_2) \times b_3$ in the expression for c in [FF, (3.4)].) In view of Remark 4.3.2, this gives constructions of all finite dimensional balanced $(1, 1)$ -FKTS's with nondegenerate skew forms. Hence, by Theorem 5.5.2, we obtain constructions of all finite dimensional simple close-to-Jordan pairs.

6. SP-GRADED KANTOR PAIRS

Assume again that \mathbb{K} is a commutative associative ring containing $\frac{1}{6}$.

6.1. SP-graded Kantor pairs. Suppose P is a trilinear pair. If G is an abelian group and $P_g = (P_g^-, P_g^+)$ for $g \in G$, where P_g^σ is a \mathbb{K} -submodule of P^σ for $g \in G$, $\sigma = \pm$, we write $P = \bigoplus_{g \in G} P_g$ to mean that $P^\sigma = \bigoplus_{g \in G} P_g^\sigma$ for $\sigma = \pm$.

We say that $P = \bigoplus_{g \in G} P_g$ is a G -grading of P if

$$\{P_g^\sigma, P_{g'}^{-\sigma}, P_{g''}^\sigma\} \subseteq P_{g-g'+g''}^\sigma$$

for $g, g', g'' \in G$, $\sigma = \pm$. In that case, each P_g is a subpair of P , and we say that P is G -graded.

If P is G -graded, we endow P^{op} with the G -grading given by

$$(P^{\text{op}})_g^\sigma = P_g^{-\sigma}.$$

We recall from Subsection 2.1 that the unmodified terms *simple* and *isomorphic* for G -graded pairs will be used in the ungraded sense, and that we have a notion of *isomorphism* for G -gradings on a trilinear pair.

If P is a Kantor pair, a *short Peirce grading* (or *SP-grading*) of P is a \mathbb{Z} -grading $P = \bigoplus_{i \in \mathbb{Z}} P_i$ such that $P_i^\sigma = 0$ for $\sigma = \pm$ and $i \neq 0, 1$. In that case we have $P = P_0 \oplus P_1$, and we call the graded pair P a *short Peirce graded* (or *SP-graded*) Kantor pair.

Any Kantor pair P has at least two SP-gradings, the *zero SP-grading* $P = P_0$ with $P_1 = 0$, and the *one SP-grading* $P = P_1$ with $P_0 = 0$. We call these two SP-gradings *trivial*.

Clearly, the opposite of an SP-graded Kantor pair is an SP-graded Kantor pair.

6.2. BC_2 -graded Lie algebras and SP-graded Kantor pairs. We now recall from [AFS, §4] the relationship between SP-graded Kantor pairs and BC_2 -graded Lie algebras.

For this purpose, suppose for the rest of this section that

$$\Delta = \Delta_{\text{BC}_2} := \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2), \pm 2\alpha_1, \pm(2\alpha_1 + \alpha_2), \pm(2\alpha_1 + 2\alpha_2)\}.$$

is the irreducible root system of type BC_2 with base $\Pi_\Delta = \{\alpha_1, \alpha_2\}$. We identify $Q_\Delta = \mathbb{Z}^2$ using the \mathbb{Z} -basis Π_Δ for Q_Δ , so any BC_2 -graded Lie algebra is a \mathbb{Z}^2 -graded Lie algebra.

If $L = \bigoplus_{(i,j) \in \mathbb{Z}^2} L_{(i,j)}$ is a \mathbb{Z}^2 -graded Lie algebra, we often write $L_{(i,j)}$ as $L_{i,j}$ for brevity. Then the *first component grading* of L is defined to be the \mathbb{Z} -grading $L = \bigoplus_{i \in \mathbb{Z}} L_{i,*}$, where $L_{i,*} = \bigoplus_{j \in \mathbb{Z}} L_{i,j}$.

Suppose that L is a BC_2 -graded Lie algebra. Then, L with its first component grading is a 5-graded Lie algebra, which therefore envelops a Kantor pair

$$P = (L_{-1,*}, L_{1,*}) = (L_{-1,0} \oplus L_{-1,-1}, L_{1,0} \oplus L_{1,1}).$$

Moreover, $P = P_0 \oplus P_1$, where

$$P_i^\sigma = L_{\sigma 1, \sigma i}$$

for $\sigma = \pm$, $i \in \mathbb{Z}$, is an SP-grading of P [AFS, §4.3]. We call P with this grading the *SP-graded Kantor pair enveloped by the BC_2 -graded Lie algebra L* , and we say that the BC_2 -graded Lie algebra L *envelops the SP-graded Kantor pair P* .

Observe that $\Delta_{\text{sh}} = \{\pm\alpha_1, \pm(\alpha_1 + \alpha_2)\}$. So if L is a BC_2 -graded Lie algebra and P is the SP-graded Kantor pair enveloped by L , then (as in the BC_1 case)

$$P^- \oplus P^+ = \bigoplus_{\alpha \in \Delta_{\text{sh}}} L_\alpha \quad \text{in } L. \quad (17)$$

Every SP-graded Kantor pair is enveloped by some BC_2 -graded Lie algebra. Indeed, if P is an SP-graded Kantor pair, then $\mathfrak{K}(P)$ has a unique BC_2 -grading, called its *standard BC_2 -grading*, such that the $\mathfrak{K}(P)$ with this grading envelops P [AFS, Prop. 4.4.1].

Note that two SP-graded Kantor pairs are graded-isomorphic if and only if the corresponding Kantor Lie algebras with their standard BC_2 -gradings are graded-isomorphic [AFS, Prop. 4.4.2].

We know that any simple SP-graded Kantor pair is enveloped by a simple BC_2 -graded Lie algebra, namely $\mathfrak{K}(P)$ with its standard BC_2 -grading. Conversely, we have the following proposition:

Proposition 6.2.1. [AFS, Prop. 4.4.2(iii)] *Let P be a nonzero SP-graded Kantor pair and suppose that L is a simple BC_2 -graded Lie algebra that envelops P . Then L is graded-isomorphic to $\mathfrak{K}(P)$ with its standard BC_2 -grading.*

Proposition 6.2.2. *If \mathbb{K} is an algebraically closed field of characteristic 0 and P is a finite dimensional simple SP-graded Kantor pair over \mathbb{K} , then $P^{\text{op}} \simeq_{\text{gr}} P$. Consequently $P_i \simeq P_i^{\text{op}}$, so the Kantor pair P_i has balanced dimension and balanced 2-dimension for $i = 0, 1$.*

Proof. By Proposition 3.1.2(iii), we can choose $\omega \in \text{Aut}(\mathfrak{K}(P))$ of period 2 such that $\omega(\mathfrak{K}(P)_{k,i}) = \mathfrak{K}(P)_{-k,-i}$ for $k, i \in \mathbb{Z}$. Then ω exchanges P_i^+ and P_i^- for $i = 0, 1$, so $(\omega|_{P^-}, \omega|_{P^+})$ is a graded-isomorphism of P onto P^{op} . Hence $P_i \simeq P_i^{\text{op}}$ for $i = 0, 1$, and the proof is complete by Lemma 4.5.1. \square

6.3. Weyl images of SP-graded Kantor pairs. Let $s_\alpha \in W_\Delta$ be the reflection through the hyperplane orthogonal to α for $\alpha \in \Delta$, and put $s_i = s_{\alpha_i}$ for $i = 1, 2$. The generators s_1 and s_2 of W_Δ satisfy $s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1 = -1$, and $\text{Aut}(\Delta) = W_\Delta = \{1, s_1, s_2, s_2 s_1, -1, -s_1, -s_2, -s_2 s_1\}$ is the dihedral group of order 8.

Let P be an SP-graded Kantor pair and let $u \in W_\Delta$. If we choose a BC_2 -graded Lie algebra L that envelops P , then ${}^u L$ is also a BC_2 -graded Lie algebra that therefore envelops an SP-graded Kantor pair which we denote by ${}^u P$. It turns out that ${}^u P$ is independent of the choice of L [AFS, Lemma 5.1.2(iv)], and we call ${}^u P$ the *u -image* (or a *Weyl image*) of P . It is clear that *Weyl images respect graded isomorphisms*; that is $P \simeq_{\text{gr}} Q \implies {}^u P \simeq_{\text{gr}} {}^u Q$. Moreover, ${}^1 P = P$, and, by (2),

$${}^{u_1}({}^{u_2} P) = {}^{u_1 u_2} P \quad \text{for } u_1, u_2 \in W_\Delta.$$

Since s_1 and s_2 generate W_Δ , the SP-graded Kantor pairs ${}^{s_1} P$ and ${}^{s_2} P$ are of particular importance. For convenience, we use the notation

$$\check{P} := {}^{s_1} P$$

and call this SP-graded Kantor pair the *reflection* of P . It is easy to check that

$$\check{P}_i^\sigma = P_i^{\pi(i)\sigma}, \quad (18)$$

for $\sigma = \pm$, $i = 0, 1$, where $\pi(0) = -$ and $\pi(1) = +$ [AFS, Prop. 5.2.1]. Moreover the σ -product $\{, \}^\sigma$ on \check{P} is given in terms of the products on P by

$$\begin{aligned} \{x_i^\sigma, y_i^{-\sigma}, z_i^\sigma\}^\sigma &= \{x_i^\sigma, y_i^{-\sigma}, z_i^\sigma\}^{\pi(i)\sigma}, \quad \{x_{1-i}^\sigma, y_{1-i}^{-\sigma}, z_i^\sigma\}^\sigma = -\{y_{1-i}^{-\sigma}, x_{1-i}^\sigma, z_i^\sigma\}^{\pi(i)\sigma}, \\ \{x_i^\sigma, y_{1-i}^{-\sigma}, z_i^\sigma\}^\sigma &= 0, \quad \text{and} \quad \{x_i^\sigma, y_{1-i}^{-\sigma}, z_{1-i}^\sigma\}^\sigma = K^{\pi(i)\sigma}(x_i^\sigma, y_{1-i}^{-\sigma})z_{1-i}^\sigma. \end{aligned}$$

for $\sigma = \pm$, $i = 0, 1$, where $x_j^\tau, y_j^\tau, z_j^\tau \in \check{P}_j^\tau$ in each case [AFS, Prop. 5.2.1]. However, we will not use these expressions in this article, but rather directly use the definition of \check{P} given above. It turns out that in general \check{P} is not isomorphic to P as an ungraded Kantor pair (as we saw in [AFS] and will see again in Example 8.3.7).

In contrast, the SP-graded Kantor pair ${}^{s_2}P$ has an easy description. We have

$${}^{s_2}P = \bar{P},$$

where \bar{P} is the SP-graded Kantor pair, called the *shift* of P that equals P as a Kantor pair and has \mathbb{Z} -grading given by $\bar{P}_i = P_{1-i}$ for $i \in \mathbb{Z}$ [AFS, (21)]. In particular, \bar{P} is isomorphic as an ungraded pair to P .

Finally, it is clear that the SP-graded pair ${}^{-1}P$ is simply P^{op} .

Proposition 6.3.1. *Suppose P is a simple SP-graded Kantor pair and $u \in W_\Delta$. Then uP is simple. Moreover, if \mathbb{K} is an algebraically closed field of characteristic 0 and P is finite dimensional, then uP has the same balanced dimension as P .*

Proof. The first statement is seen in [AFS, Prop. 4.1.4]. For the second statement, it suffices to prove that \bar{P} and \check{P} have the same balanced dimension as P (since $W_\Delta = \langle s_1, s_2 \rangle$). This is clear for \bar{P} ; while for \check{P} it follows from (18) and the fact that P_0 has balanced dimension by Proposition 6.2.2. \square

Remark 6.3.2. Under the assumptions of the second statement of the proposition, it is clear that the shift \bar{P} of an SP-graded P has the same balanced 2-dimension as P . However that is not true for the reflection \check{P} of P (see Example 8.3.7 below).

7. CLASSIFICATION OF FINITE DIMENSIONAL SIMPLE SP-GRADED KANTOR PAIRS

For the rest of this article we assume that \mathbb{K} is an algebraically closed field of characteristic 0, and Π is the connected Dynkin diagram of type X_n .

We also assume \mathcal{G} is a finite dimensional simple Lie algebra over \mathbb{K} of type X_n with Cartan subalgebra \mathcal{H} and root system $\Sigma = \Sigma(\mathcal{G}, \mathcal{H})$ relative to \mathcal{H} . Let Π_Σ be a base for this root system, in which case there exists a diagram isomorphism $\iota : \Pi \rightarrow \Pi_\Sigma$, which we use to identify $\Pi = \Pi_\Sigma$.

Also, as in Section 5, μ^+ is the highest root of Σ relative to Π , $\mu^- = -\mu^+$ and $\tilde{\Pi} = \Pi \cup \{\mu^-\}$ is the extended Dynkin diagram for Π . If Y is a non-empty subset of $\tilde{\Pi}$, then Y is the disjoint union of its connected components; and if $\lambda \in Y$, we use the notation $\text{comp}(Y, \lambda)$ for the connected component of Y containing λ .

We further assume that

$$\Delta = \Delta_{\text{BC}_2} := \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2), \pm 2\alpha_1, \pm(2\alpha_1 + \alpha_2), \pm(2\alpha_1 + 2\alpha_2)\}$$

is the irreducible root system of type BC_2 with base $\Pi_\Delta = \{\alpha_1, \alpha_2\}$.

We identify $Q_\Delta = \mathbb{Z}^2$ using the \mathbb{Z} -basis Π_Δ for Q_Δ . We then have the identification

$$\mathrm{Hom}(Q_\Sigma, Q_\Delta) = \mathrm{Hom}(Q_\Sigma, \mathbb{Z}^2) = \mathrm{Hom}(Q_\Sigma, \mathbb{Z})^2,$$

where in the last equality (ρ_1, ρ_2) in $\mathrm{Hom}(Q_\Sigma, \mathbb{Z})^2$ is identified with the element of $\mathrm{Hom}(Q_\Sigma, \mathbb{Z}^2)$ given by $\mu \mapsto (\rho_1(\mu), \rho_2(\mu))$.

7.1. SP-admissible pairs of subsets of Π . To emphasize the role of BC_2 here, we will write $\mathrm{Hom}(\Sigma, \Delta)$, $\mathrm{Hom}_{\mathrm{sh}}(\Sigma, \Delta)$, $\mathrm{Hom}_{\mathrm{p}}(\Sigma, \Delta)$ and $\mathrm{Hom}_{\mathrm{psh}}(\Sigma, \Delta)$ respectively as $\mathrm{Hom}(\Sigma, \mathrm{BC}_2)$, $\mathrm{Hom}_{\mathrm{sh}}(\Sigma, \mathrm{BC}_2)$, $\mathrm{Hom}_{\mathrm{p}}(\Sigma, \mathrm{BC}_2)$ and $\mathrm{Hom}_{\mathrm{psh}}(\Sigma, \mathrm{BC}_2)$. So, setting $\square_2 := \{0, 1, 2\} \times \{0, 1, 2\}$, we have

$$\mathrm{Hom}(\Sigma, \mathrm{BC}_2) = \{\rho \in \mathrm{Hom}(Q_\Sigma, \mathbb{Z}^2) :$$

$$\rho(\Sigma) \subseteq \square_2 \cup (-\square_2), (1, 2) \notin \rho(\Sigma), (0, 2) \notin \rho(\Sigma)\}, \quad (19)$$

$$\mathrm{Hom}_{\mathrm{sh}}(\Sigma, \mathrm{BC}_2) = \{(\rho_1, \rho_2) \in \mathrm{Hom}(\Sigma, \mathrm{BC}_2) : 1 \in \rho_1(\Sigma)\}. \quad (20)$$

If (S, T) is a pair of subsets of Π , we use the notation

$$\chi_{(S, T)} := (\chi_S, \chi_T) \in \mathrm{Hom}(Q_\Sigma, \mathbb{Z}^2)$$

and say (S, T) is *SP-admissible* if $\chi_{(S, T)} \in \mathrm{Hom}_{\mathrm{sh}}(\Sigma, \mathrm{BC}_2)$ (or equivalently $\chi_{(S, T)} \in \mathrm{Hom}_{\mathrm{psh}}(\Sigma, \mathrm{BC}_2)$). We let

$\mathrm{SPA}(\Pi)$ = the set of all SP-admissible pairs of subsets of Π .

Remark 7.1.1. Just as in Remark 5.1.1, the set $\mathrm{SPA}(\Pi)$ is well-defined.

Proposition 7.1.2. *The map $(S, T) \mapsto \chi_{(S, T)}$ is a bijection of the set $\mathrm{SPA}(\Pi)$ onto the set $\mathrm{Hom}_{\mathrm{psh}}(\Sigma, \mathrm{BC}_2)$.*

Proof. All but surjectivity is clear. For surjectivity, suppose that $\rho = (\rho_1, \rho_2) \in \mathrm{Hom}_{\mathrm{psh}}(\Sigma, \mathrm{BC}_2)$. Then, by (19) and (20), $\rho_1(\Sigma) \subseteq \{0, \pm 1, \pm 2\}$ and $1 \in \rho_1(\Sigma)$. Hence, since $\rho_1(\alpha) \geq 0$ for $\alpha \in \Sigma^+$, we have $\rho_1 = \chi_S$ for some $S \subseteq \Pi$ by Proposition 5.1.2. Also, if $1 \in \rho_2(\Sigma)$, we have $\rho_2 = \chi_T$ for some $T \subseteq \Pi$ by the same argument. So we can assume that $1 \notin \rho_2(\Sigma)$. Hence, by (19), $\rho(\Sigma) \subseteq \{(0, 0), \pm(1, 0), \pm(2, 0), \pm(2, 2)\}$. Therefore, $\mathcal{G}(\rho)_{-1,*} + \mathcal{G}(\rho)_{1,*} \subseteq \mathcal{G}(\rho)_{*,0}$ (using the notation of Section 6.2). But the left hand side of this inclusion is nonzero by Lemma 3.2.1, so this space generates \mathcal{G} as an algebra by (7). Therefore $\mathcal{G}(\rho) = \mathcal{G}(\rho)_{*,0}$, so $\rho_2(\Sigma) = 0$. Thus $\rho_2 = 0 = \chi_\emptyset$. \square

Lemma 7.1.3. *Let $\rho = (\rho_1, \rho_2) \in \mathrm{Hom}(Q_\Sigma, \mathbb{Z}^2)$. Then $\rho \in \mathrm{Hom}_{\mathrm{sh}}(\Sigma, \mathrm{BC}_2)$ if and only if $\rho(\Sigma) \subseteq \square_2 \cup (-\square_2)$, $(1, 2) \notin \rho(\Sigma)$ and $1 \in \rho_1(\Sigma)$.*

Proof. In view of (19), we only need to prove “ \Leftarrow ”. Let $L = \mathcal{G}(\rho)$, which is a \mathbb{Z}^2 -graded Lie algebra with $\mathrm{supp}_{\mathbb{Z}^2}(L) = \rho(\Sigma \cup \{0\})$. By our assumptions, the first component grading $L = \bigoplus_{i \in \mathbb{Z}} L_{i,*}$ of L is a 5-grading with $L_{-1,*} + L_{1,*} \neq 0$. Hence, by (7), each element in L has the form $x + \sum [y_j, z_j]$, where $x, y_j, z_j \in L_{-1,*} + L_{1,*}$. Therefore each element of $\mathrm{supp}_{\mathbb{Z}^2}(L)$ is either an element in $\mathrm{supp}_{\mathbb{Z}^2}(L_{-1,*} + L_{1,*})$ or it is a sum of two such elements. But by our assumptions $\mathrm{supp}_{\mathbb{Z}^2}(L_{-1,*} + L_{1,*}) \subseteq \{\pm(1, 0), \pm(1, 1)\}$. Hence $(0, 2) \notin \mathrm{supp}_{\mathbb{Z}^2}(L)$ as needed. \square

Lemma 7.1.4. *If $\mu \in \Sigma$, then $\{\mu^-\} \cup \mathrm{supp}_\Pi\{\mu^+ - \mu\}$ is a connected subset of $\tilde{\Pi}$.*

Proof. Let $S(\mu) = \{\mu^-\} \cup \text{supp}_\Pi\{\mu^+ - \mu\}$ for $\mu \in \Sigma$. We prove that $S(\mu)$ is connected by induction on the non-negative integer $\text{ht}_\Pi(\mu^+ - \mu)$. First, if $\text{ht}_\Pi(\mu^+ - \mu) = 0$, then $\mu = \mu^+$ and $S(\mu) = \{\mu^-\}$ is connected. Suppose that $\text{ht}_\Pi(\mu^+ - \mu) > 0$. Then $\mu \neq \mu^+$. Furthermore, we can assume that $\mu \notin -\Pi$, since otherwise $S(\mu) = \tilde{\Pi}$ is connected. So there exists $\lambda \in \Pi$ with

$$\nu = \mu + \lambda \in \Sigma.$$

Then, by the induction hypothesis, $S(\nu)$ is connected. Further, $S(\mu) = S(\nu) \cup \{\lambda\}$. So it remains to show that $\langle S(\nu), \lambda \rangle \neq 0$. For this we can assume that $\lambda \notin S(\nu)$, so $\chi_\lambda(\nu) = \chi_\lambda(\mu^+)$. Hence $\nu + \lambda \notin \Sigma$. Thus, since $\nu - \lambda \in \Sigma$, we have $\langle \nu, \lambda \rangle \neq 0$. But then, since ν lies in the group generated by $S(\nu)$, we have $\langle S(\nu), \lambda \rangle \neq 0$. \square

We have the following characterization of SP-admissible pairs of subsets of Π .

Proposition 7.1.5. *Suppose that $S, T \subseteq \Pi$. Then the following are equivalent:*

- (a) $(S, T) \in \text{SPA}(\Pi)$.
- (b) $S \neq \emptyset$ and $\chi_{(S,T)} \in \text{Hom}(\Sigma, \text{BC}_2)$
- (c) $\chi_S(\mu^+) \in \{1, 2\}$; $\chi_T(\mu^+) \in \{0, 1, 2\}$; and if $\chi_T(\mu^+) = 2$, then $\chi_S(\mu^+) = 2$ and $\text{comp}(\tilde{\Pi} \setminus T, \mu^-) \cap S = \emptyset$.

Proof. The equivalence of (a) and (b) is clear, so we only consider the equivalence of (b) and (c). For this we can assume that $\chi_S(\mu^+) \in \{1, 2\}$ and $\chi_T(\mu^+) \in \{0, 1, 2\}$ (since these statements hold if either (b) or (c) is assumed). Also, if $\chi_T(\mu^+) = 0$ or 1, then $\chi_T(\mu) \neq \pm 2$ for $\mu \in \Sigma$, so (b) and (c) are each true. Thus we can suppose $\chi_T(\mu^+) = 2$. Then if $\chi_S(\mu^+) = 1$, statements (b) and (c) are each false. So we can suppose $\chi_S(\mu^+) = 2$. It remains to show that $(S, T) \in \text{SPA}(\Pi)$ if and only if $\text{comp}(\tilde{\Pi} \setminus T, \mu^-) \cap S = \emptyset$. We establish the contrapositives of these implications.

Suppose that $\text{comp}(\tilde{\Pi} \setminus T, \mu^-) \cap S \neq \emptyset$. Then there is a path $\lambda_0, \lambda_1, \dots, \lambda_r$ in $\tilde{\Pi}$ which does not pass through T with $r \geq 1$, $\lambda_0 = \mu^-$, $\lambda_r \in S$. We can shorten this path if necessary to assume that the λ_i 's are distinct and that $\lambda_i \in \Pi \setminus S$ for $1 \leq i \leq r-1$. Then $P := \{\lambda_0, \lambda_1, \dots, \lambda_r\}$ is a non-empty, proper and connected subset of $\tilde{\Pi}$, so the diagram for P is the diagram of an irreducible reduced finite root system [Kac, Prop. 4.7(c)]. Hence $\mu_P := \sum_{i=0}^r \lambda_i \in \Sigma$. So

$$-\mu_P = \mu^+ - \lambda_1 - \dots - \lambda_r \in \Sigma^+.$$

Since $\lambda_i \notin T$ for $1 \leq i \leq r$, we have $\chi_T(-\mu_P) = \chi_T(\mu^+) = 2$. Also, since $\lambda_i \notin S$ for $1 \leq i \leq r-1$, we have $\chi_S(-\mu_P) = \chi_S(\mu^+) - \chi_S(\lambda_r) = 2 - 1 = 1$. Thus $(S, T) \notin \text{SPA}(\Pi)$.

Conversely, suppose that $(S, T) \notin \text{SPA}(\Pi)$. Then, by Lemma 7.1.3, there exists $\mu \in \Sigma^+$ with $(\chi_S(\mu), \chi_T(\mu)) = (1, 2)$. Let $S(\mu) = \{\mu^-\} \cup \text{supp}_\Pi\{\mu^+ - \mu\}$. Now $\chi_T(\mu^+ - \mu) = 2 - 2 = 0$, so $T \cap S(\mu) = \emptyset$. Also, $\chi_S(\mu^+ - \mu) = 2 - 1 = 1$, so $S \cap S(\mu) \neq \emptyset$. Since $S(\mu)$ is connected by Lemma 7.1.4, $S(\mu) \subseteq \text{comp}(\tilde{\Pi} \setminus T, \mu^-)$, so $\text{comp}(\tilde{\Pi} \setminus T, \mu^-) \cap S \neq \emptyset$. \square

It follows from Propositions 7.1.5(c) and 5.1.3(c) that if $(S, T) \in \text{SPA}(\Pi)$, then S is Kantor-admissible and T is either empty or Kantor-admissible.

We define a right action of $\text{Aut}(\Pi)$ on the set of pairs of subsets of Π by

$$(S, T) \cdot \varphi := (S \cdot \varphi, T \cdot \varphi) = (\varphi^{-1}(S), \varphi^{-1}(T)).$$

Now, by (10), we have

$$\chi_{(S,T)} \cdot \varphi = \chi_{(S,T) \cdot \varphi} \quad (21)$$

for $S, T \subseteq \Pi$, $\varphi \in \text{Aut}(\Pi)$. Using this and the fact that $\text{Hom}_{\text{sh}}(\Sigma, \text{BC}_2)$ is stabilized by the right action of $\text{Aut}(\Pi)$, we see that the $\text{SPA}(\Pi)$ is stabilized by the right action of $\text{Aut}(\Pi)$. So we have a right action \cdot of $\text{Aut}(\Pi)$ on $\text{SPA}(\Pi)$.

7.2. Classification.

Construction 7.2.1. (The SP-graded Kantor pair $\mathfrak{P}(\Pi; S, T)$) Suppose $(S, T) \in \text{SPA}(\Pi)$, in which case $\chi_{(S, T)} \in \text{Hom}_{\text{sh}}(\Sigma, \text{BC}_2)$. Then, by Lemma 3.2.1, $\mathcal{G}(\chi_{(S, T)})$ is a BC_2 -graded Lie algebra with $\mathcal{G}(\chi_{(S, T)})_\alpha \neq 0$ for some $\alpha \in \Delta_{\text{sh}}$. Let $\mathfrak{P}(\Pi; S, T)$ be the SP-graded Kantor pair enveloped by $\mathcal{G}(\chi_{(S, T)})$ (see Section 6.2). Note that $\mathfrak{P}(\Pi; S, T)$ is nonzero by (17). Explicitly $\mathfrak{P}(\Pi; S, T)$ is the Kantor pair $\mathfrak{P}(\Pi; S)$ (see Construction 5.2.1) with the SP-grading given by

$$\mathfrak{P}(\Pi; S)_{\sigma i}^\sigma = \mathcal{G}(\chi_{(S, T)})_{\sigma 1, \sigma i} = \sum_{\mu \in Q_\Sigma, \chi_S(\mu) = \sigma 1, \chi_T(\mu) = \sigma i} \mathcal{G}_\mu \quad (22)$$

for $\sigma = \pm$ and $i = 0, 1$. We call this grading the *SP-grading of $\mathfrak{P}(\Pi; S)$ determined by T* .

Remark 7.2.2. Just as in Remark 5.2.2, $\mathfrak{P}(\Pi; S, T)$ is well-defined up to graded-isomorphism.

Again since Π is fixed in our discussion we will usually write $\mathfrak{P}(\Pi; S, T)$ as $\mathfrak{P}(S, T)$.

Remark 7.2.3. Let $(S, T) \in \text{SPA}(\Pi)$.

(i) By Proposition 6.2.1, $\mathcal{G}(\chi_{(S, T)})$ is graded-isomorphic to $\mathfrak{K}(\mathfrak{P}(S, T))$ with its standard BC_2 -grading.

(ii) If $i = 0, 1$, $\mathfrak{P}(S, T)_i$ has balanced dimension by Proposition 6.2.2. Moreover, this balanced dimension equals $\dim(\mathcal{G}(\chi_{(S, T)})_{1, i})$ by (i), which equals

$$|\{\mu \in \Sigma : \chi_S(\mu) = 1, \chi_T(\mu) = i\}|.$$

We have the following classification of simple SP-graded Kantor pairs of type X_n .

Theorem 7.2.4. Suppose that Π is the connected Dynkin diagram of type X_n .

- (i) If $(S, T) \in \text{SPA}(\Pi)$, then $\mathfrak{P}(S, T)$ is a finite dimensional simple SP-graded Kantor pair of type X_n .
- (ii) If P is a finite dimensional simple SP-graded Kantor pair of type X_n , then P is graded-isomorphic to $\mathfrak{P}(S, T)$ for some $(S, T) \in \text{SPA}(\Pi)$.
- (iii) If $(S, T), (S', T') \in \text{SPA}(\Pi)$, then the Kantor pairs $\mathfrak{P}(S, T)$ and $\mathfrak{P}(S', T')$ are graded-isomorphic if and only if (S, T) and (S', T') are in the same orbit in $\text{SPA}(\Pi)$ under the right action of $\text{Aut}(\Pi)$.

Proof. (i): This follows from Proposition 4.6.3.

(ii): $\mathfrak{K}(P)$ is simple of type X_n , so there is an isomorphism $\eta : \mathfrak{K}(P) \rightarrow \mathcal{G}$. We use η to transport the standard BC_2 -grading of $\mathfrak{K}(P)$ to \mathcal{G} , so η is a BC_2 -graded isomorphism. Next by Theorem 3.2.2(i) we can assume that the BC_2 -grading of \mathcal{G} is the ρ -grading for some $\rho \in \text{Hom}_{\text{p}}(\Sigma, \text{BC}_2)$. Now $\mathfrak{K}(P)_\alpha \neq 0$ for some $\alpha \in \Delta_{\text{sh}}$ by (17), and thus $\mathcal{G}(\rho)_\alpha \neq 0$ for some $\alpha \in \Delta_{\text{sh}}$. So by Lemma 3.2.1(ii), $\rho \in \text{Hom}_{\text{psh}}(\Sigma, \text{BC}_2)$, and hence by Proposition 7.1.2, $\rho = \chi_{(S, T)}$ for some $(S, T) \in \text{SPA}(\Pi)$. Thus, under the restriction of η , $P \simeq_{\text{gr}} \mathfrak{P}(S, T)$.

(iii): Let $P = \mathfrak{P}(S, T)$ and $P' = \mathfrak{P}(S', T')$; and let $\rho = \chi_{(S, T)}$ and $\rho' = \chi_{(S', T')}$ in $\text{Hom}_{\text{psh}}(\Sigma, \text{BC}_2)$. Then by Theorem 3.2.2(ii) and (21), we know that $\mathcal{G}(\rho) \simeq_{\text{gr}} \mathcal{G}(\rho')$ if and only if (S, T) and (S', T') are in the same orbit in $\text{SPA}(\Pi)$ under the right

action of $\text{Aut}(\Pi)$. So it remains to show that $P \simeq_{\text{gr}} P'$ if and only if $\mathcal{G}(\rho) \simeq_{\text{gr}} \mathcal{G}(\rho')$. The implication “ \Leftarrow ” in this statement is clear. To prove the converse, suppose that $P \simeq_{\text{gr}} P'$. Then, as noted in Subsection 6.2, $\mathfrak{K}(P) \simeq_{\text{gr}} \mathfrak{K}(P')$; so, by Remark 7.2.3(i), $\mathcal{G}(\rho) \simeq_{\text{gr}} \mathcal{G}(\rho')$. \square

Remark 7.2.5. We note that if we take $T = \emptyset$ everywhere in Theorem 7.2.4, we obtain the classification Theorem 5.2.4.

As a corollary, we obtain the following classification up to isomorphism of the SP-gradings on a fixed simple Kantor pair.

Corollary 7.2.6. *Suppose $S \in \text{KA}(\Pi)$. Any SP-grading on $\mathfrak{P}(S)$ is isomorphic to the SP-grading determined by T for some subset T of Π such that $(S, T) \in \text{SPA}(\Pi)$. Also, for two such subsets T and T' of Π , the SP-gradings of $\mathfrak{P}(S)$ determined by T and T' are isomorphic if and only if (S, T) and (S, T') are in the same orbit in $\text{SPA}(\Pi)$ under the right action of $\text{Aut}(\Pi)$.*

If $S \in \text{KA}(\Pi)$, then, by Proposition 7.1.5(c), (S, \emptyset) and (S, S) are in $\text{SPA}(\Pi)$. Moreover, the SP-gradings of $\mathfrak{P}(S)$ determined by \emptyset and S are respectively the zero SP-grading and the one SP-grading of $\mathfrak{P}(S)$. Of course *our main interest is in non-trivial gradings of $\mathfrak{P}(S)$, which occur in Corollary 7.2.6 when T is not equal to \emptyset or S .*

7.3. The close-to-Jordan case. The classification of SP-gradings has a particularly simple description for close-to-Jordan pairs.

Theorem 7.3.1. *Suppose $S \in \text{KA}(\Pi)$ and $P = \mathfrak{P}(S)$ is close-to-Jordan. If $T \subseteq \Pi$, then $(S, T) \in \text{SPA}(\Pi)$ if and only if*

$$T = \emptyset, T = S \text{ or } T = \{\lambda\} \text{ for some } \lambda \in \Pi \text{ with } \chi_\lambda(\mu^+) = 1. \quad (23)$$

Hence the SP-gradings of $\mathfrak{P}(S)$ are, up to isomorphism, precisely the SP-gradings determined by subsets of Π of the form (23). Finally, if T and T' are subsets of Π of the form (23), then T and T' determine isomorphic SP-gradings of $\mathfrak{P}(S)$ if and only if there exists $\varphi \in \text{Aut}(\Pi)$ such that $T' = T \cdot \varphi$.

Proof. Recall that $X_n \neq A_1$, S is the set of nodes of Π that are adjacent to μ^- in $\tilde{\Pi}$ and $\chi_S(\mu^+) = 2$ (see Theorem 5.4.1). If T of the form (23), then $(S, T) \in \text{SPA}(\Pi)$ by Proposition 7.1.5(c). For the converse, suppose $T \subseteq \Pi$ with $(S, T) \in \text{SPA}(\Pi)$, and suppose $T \neq \emptyset$ and $T \neq S$. Then it suffices to show that $\chi_T(\mu^+) = 1$, so we suppose the contrary. Hence $\mu_T(\mu^+) = 2$. But if $\nu \in S \setminus T$, then μ^-, ν is a path in $\tilde{\Pi} \setminus T$, so $\nu \in \text{comp}(\tilde{\Pi}, \mu^-) \cap S$, which is empty by Proposition 7.1.5(c). Therefore $S \subseteq T$. So since $\chi_T(\mu^+) = 2 = \chi_S(\mu^+)$, we have $T = S$. With this contradiction we have proved the first statement. The second and third statements now follow by Corollary 7.2.6 and the fact that each $\varphi \in \text{Aut}(\Pi)$ fixes μ^- . \square

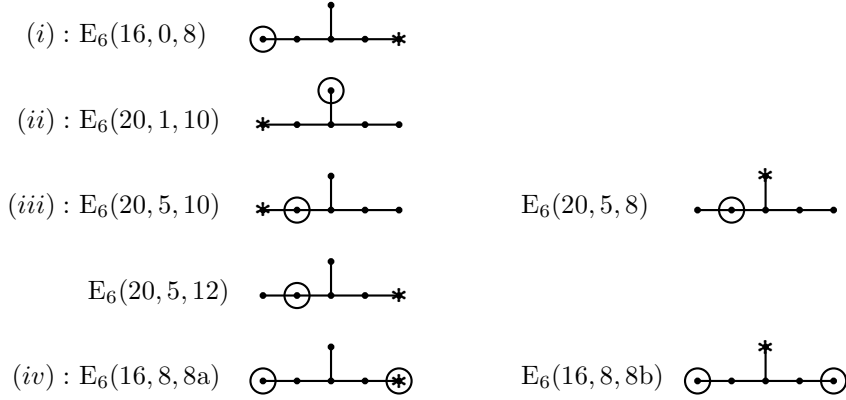
Looking at the explicit expression for μ^+ for each type, we see the following:

Corollary 7.3.2. *Suppose that P is the simple close-to-Jordan Kantor pair of type $X_n \neq A_1$. If $X_n = G_2, F_4$ or E_8 , then P does not have a non-trivial SP-grading. For the other types, the number of isomorphism classes of non-trivial SP-gradings of P is $\lfloor \frac{n+1}{2} \rfloor$ if $X_n = A_n (n \geq 2)$, 2 if $X_n = D_n (n \geq 5)$ and 1 if $X_n = B_n (n \geq 2)$, $C_n (n \geq 3)$, D_4 , E_6 or E_7 .*

See Example 7.4.1(ii) below for an illustration of Theorem 7.3.1 and Corollary 7.3.2 in type E_6 .

7.4. Marked Dynkin diagrams for SP-gradings. We represent an element $(S, T) \in \text{SPA}(\Pi)$ by the Dynkin diagram for Π with the nodes in S marked with a circle and the nodes in T marked with an asterisk; and we use the same marked Dynkin diagram to represent the SP-graded Kantor pair $\mathfrak{P}(S, T)$ as well as the SP-grading of $\mathfrak{P}(S)$ determined by T .

Example 7.4.1. (Type E_6) Suppose that $\Pi = \{\mu_1, \dots, \mu_6\}$ is of type E_6 with Dynkin diagram as in Example 5.3.1. Recall that in Example 5.3.1 we saw that (up to the right action of $\text{Aut}(\Pi)$) there are four Kantor-admissible subsets S of Π : (i) $\{\mu_1\}$, (ii) $\{\mu_6\}$, (iii) $\{\mu_2\}$, and (iv) $\{\mu_1, \mu_5\}$. We now use Proposition 7.1.5(c) and the extended diagram (11) for E_6 to list for each choice of S the marked diagrams that represent, up to the right action of $\text{Aut}(\Pi)$, the SP-admissible pairs of the form (S, T) with $T \neq \emptyset$ and $T \neq S$:



Hence, for each S , Corollary 7.2.6 tells us that the non-trivial SP-gradings on $\mathfrak{P}(S)$ are represented up to isomorphism by the listed marked diagrams. Thus, up to graded-isomorphism, each simple SP-graded Kantor pair of type E_6 with non-trivial grading is represented by exactly one of the above seven marked diagrams.

Note that each marked diagram representing (S, T) and $\mathfrak{P}(S, T)$ in (i), (ii) and (iii) is labelled as $E_6(d, e, f)$, where d is the balanced dimension of $\mathfrak{P}(S, T)$ and e is the balanced 2-dimension of $\mathfrak{P}(S, T)$ as in Example 5.3.1, and where f is the balanced dimension of $\mathfrak{P}(S, T)_1$. (f was computed using Remark 7.2.3(ii) and a list of roots in Σ [B, Plate V].) In (iv), we have used the notations $E_6(16, 8, 8a)$ and $E_6(16, 8, 8b)$ because there are two graded pairs with parameters 16, 8, 8 that are not graded-isomorphic. In any of the cases, we will sometimes use the label for the SP-graded Kantor pair $\mathfrak{P}(S, T)$ itself.

For each of the other types X_n , it is not difficult using the same method to write down marked Dynkin diagrams representing up to graded-isomorphism the simple SP-graded Kantor pairs of type X_n with non-trivial gradings. We leave this to the interested reader as an exercise.

8. WEYL IMAGES OF FINITE DIMENSIONAL SIMPLE SP-GRADED KANTOR PAIRS

We continue with the assumptions and notation of Section 7. In this section (see Theorems 8.3.1 and 8.3.2), we compute the marked Dynkin diagram that represents the SP-graded Kantor pair ${}^u\mathfrak{P}(S, T)$ for each $(S, T) \in \text{SPA}(\Pi)$ and each $u \in W_\Delta$.

In view of Theorem 7.2.4, this computes all Weyl images of all finite dimensional simple SP-graded Kantor pairs.

8.1. The maps w_X , σ_X and \tilde{w}_X . Suppose that $X \subseteq \Pi$.

Let

$$E_X = \text{span}_{\mathbb{R}}(X), \quad \Sigma_X = \Sigma \cap E_X \quad \text{and} \quad \Sigma_X^+ = \Sigma^+ \cap E_X.$$

If $X \neq \emptyset$, then Σ_X is a root system with base X in the Euclidean space E_X , and Σ_X^+ is the set of positive roots in Σ_X relative to X . We then use the simplified notation

$$Q_X = Q_{\Sigma_X} \quad \text{and} \quad W_X = W_{\Sigma_X}$$

respectively for the root lattice and Weyl group of Σ_X . If $X = \emptyset$, then $E_X = \{0\}$, $\Sigma_X = \emptyset$, $\Sigma_X^+ = \emptyset$, and we set $Q_X = \{0\}$ and $W_X = \{1\}$.

If $X \neq \emptyset$, we let w_X be the unique element of W_X such that

$$w_X(\Sigma_X^+) = -\Sigma_X^+.$$

Equivalently, w_X is the longest element of W_X relative to the base X [B, VI, §1.6, Cor. 3]. Clearly $-w_X$ stabilizes X and we set

$$\sigma_X = -w_X \in \text{Aut}(X),$$

where recall that $\text{Aut}(X)$ is the group of diagram automorphisms of X . If $X = \emptyset$, we adopt the conventions that $w_X = 1$ and $\sigma_X = 1$. Evidently in all cases $w_X^2 = 1$ and $\sigma_X^2 = 1$.

If $X \neq \emptyset$, it is well known that the map σ_X can be read from the Dynkin diagram for X . Indeed, if X is connected, then σ_X is the nontrivial diagram automorphism of X if X has type A_n with $n \geq 2$, D_n with n odd ≥ 5 , or E_6 ; and $\sigma_X = 1$ for all other types [B, VI, §4.5–§4.13]. Furthermore, if X has connected components X_i , $1 \leq i \leq r$, then σ_X stabilizes each X_i and $\sigma_X|_{X_i} = \sigma_{X_i}$.

If $w \in W_X$, then w extends uniquely to an isometry \tilde{w} of E with $\tilde{w}(\tau) = \tau$ for $(\tau, E_X) = 0$. It is clear that the map $\varphi \mapsto \tilde{\varphi}$ is a monomorphism of W_X into W_Σ .

We will abuse notation and write $\widetilde{w_X}$ as \tilde{w}_X . Note that if $\lambda \in E$, we have

$$\tilde{w}_X(\lambda) \in \lambda + Q_X. \tag{24}$$

8.2. The left action $*$ of W_Δ on $\text{SPA}(\Pi)$. Recall from Section 1.4 that we have left action $*$ of $W_\Delta = \text{Aut}(\Delta)$ on $\text{Hom}_{\text{psh}}(\Sigma, \text{BC}_2)$. We use the bijection $(S, T) \mapsto \chi_{(S, T)}$ in Proposition 7.1.2 to transfer this action to a *left action $*$ of W_Δ on $\text{SPA}(\Pi)$* . So by definition we have for $u \in W_\Delta$ and $(S, T) \in \text{SPA}(\Pi)$ that

$$\chi_{u*(S, T)} = u * \chi_{(S, T)}. \tag{25}$$

Using (21) and (25), we see that the left action $*$ of W_Δ on $\text{SPA}(\Pi)$ commutes with the right action \cdot of $\text{Aut}(\Pi)$ on the same set.

8.3. Weyl images of $\mathfrak{P}(S, T)$.

Theorem 8.3.1. *Suppose that $(S, T) \in \text{SPA}(\Pi)$. Then ${}^u\mathfrak{P}(S, T) \simeq_{\text{gr}} \mathfrak{P}(u * (S, T))$ for $u \in W_\Delta$.*

Proof. By definition of $*$, we have $u * \chi_{(S, T)} = u \cdot \chi_{(S, T)} \cdot w$ for some $w \in W_\Sigma$. Then

$$\begin{aligned} {}^u\mathcal{G}(\chi_{(S, T)}) &= \mathcal{G}(u \cdot \chi_{(S, T)}) && \text{by (4)} \\ &\simeq_{\text{gr}} \mathcal{G}(u \cdot \chi_{(S, T)} \cdot w) && \text{by Proposition 3.1.2(ii)} \\ &= \mathcal{G}(u * \chi_{(S, T)}) = \mathcal{G}(\chi_{u*(S, T)}) && \text{by (25).} \end{aligned}$$

Since ${}^u\mathcal{G}(\chi_{(S,T)})$ and $\mathcal{G}(\chi_{u*(S,T)})$ determine ${}^u\mathfrak{P}(S, T)$ and $\mathfrak{P}(u*(S, T))$ respectively, we have our conclusion. \square

Because of Theorem 8.3.1. it is important to be able to compute $u*(S, T)$ for $u \in W_\Delta$, $(S, T) \in \text{SPA}(\Pi)$. We do this in the next theorem, where there is a case that requires special treatment, namely the case when:

$$\begin{aligned} &\Pi \text{ has type } A_n, n \geq 3, \text{ and there exists distinct } \lambda, \lambda', \mu \in \Pi \\ &\text{such that } S = \{\lambda, \lambda'\}, T = \{\mu\}, \text{ and } \lambda' \text{ lies between } \lambda \text{ and } \mu \text{ on the Dynkin diagram for } \Pi. \end{aligned} \quad (\text{ex})$$

Recall from Section 6.3 that $W_\Delta = \{1, s_1, s_2, s_2s_1, -1, -s_1, -s_2, -s_2s_1\}$.

Theorem 8.3.2. *Suppose $(S, T) \in \text{SPA}(\Pi)$. Then the elements $u*(S, T)$ for $u \in W_\Delta$ are given in the following table*

u	$u*(S, T)$
1	(S, T)
s_1	(\check{S}, T)
s_2	(S, \bar{T})
s_2s_1	$(\check{S}, \bar{T} \cdot \sigma_\Pi)$
-1	$(S \cdot \sigma_\Pi, T \cdot \sigma_\Pi)$
- s_1	$(\check{S} \cdot \sigma_\Pi, T \cdot \sigma_\Pi)$
- s_2	$(S \cdot \sigma_\Pi, \bar{T} \cdot \sigma_\Pi)$
- s_2s_1	$(\check{S} \cdot \sigma_\Pi, \bar{T})$

(26)

where

$$\check{S} = \sigma_{\Pi \setminus T}(S \setminus T) \cup \{\lambda \in T : \chi_{S \setminus T}(\tilde{w}_{\Pi \setminus T}(\lambda)) + \chi_{S \cap T}(\lambda) = 1\} \quad (27)$$

and

$$\bar{T} = \sigma_{\Pi \setminus S}(T \setminus S) \cup \{\lambda \in S \setminus T : \text{supp}_\Pi(\tilde{w}_{\Pi \setminus S}(\lambda)) \cap (T \setminus S) = \emptyset\}. \quad (28)$$

(See Remark 8.3.4 below about the notation \check{S} and \bar{T} .) Moreover, we have the following expressions, which allow us to read \check{S} and \bar{T} directly from the marked Dynkin diagram representing (S, T) together with the expression for the highest root μ^+ in Σ :

$$\check{S} := \begin{cases} S & \text{if } \chi_{S \setminus T}(\mu^+) = 0, \\ \sigma_{\Pi \setminus T}(S \setminus T) \cup (T \setminus S) & \text{if } \chi_{S \setminus T}(\mu^+) = 1, \\ \sigma_{\Pi \setminus T}(S \setminus T) & \text{if } \chi_{S \setminus T}(\mu^+) = 2, \end{cases} \quad (29)$$

and

$$\bar{T} := \begin{cases} S \setminus T & \text{if } T \subseteq S, \\ \{\sigma_{\Pi \setminus S}(\mu), \lambda\} & \text{if (ex) holds,} \\ \sigma_{\Pi \setminus S}(T \setminus S) & \text{otherwise.} \end{cases} \quad (30)$$

Proof. We postpone the proofs of (29) and (30) until Section 8.4, since these proofs require some additional information about the Weyl group elements $\tilde{w}_{\Pi \setminus S}$ and $\tilde{w}_{\Pi \setminus T}$. We prove the first statement here.

If $\lambda \in \Pi$, we have

$$((-1) \cdot \chi_{(S,T)} \cdot w_\Pi)(\lambda) = -(\chi_S(w_\Pi(\lambda)), \chi_T(w_\Pi(\lambda))) = (\chi_S(\sigma_\Pi(\lambda)), \chi_T(\sigma_\Pi(\lambda))),$$

which has non-negative entries. Hence, $(-1) * \chi_{(S,T)} = (-1) \cdot \chi_{(S,T)} \cdot w_\Pi = \chi_{(S,T)} \cdot \sigma_\Pi = \chi_{(S,T) \cdot \sigma_\Pi}$ using (21). So using (25), $(-1) * (S, T) = (S, T) \cdot \sigma_\Pi = (S \cdot \sigma_\Pi, T \cdot \sigma_\Pi)$. This establishes row 5 (not counting the header row) of (26), and we see that rows 6, 7 and 8 follow respectively from rows 2, 3 and 4. Thus it is sufficient to establish rows 2, 3 and 4.

Row 2: Let $\rho = (\rho_1, \rho_2) = s_1 \cdot \chi_{(S,T)} \cdot \tilde{w}_{\Pi \setminus T} \in \text{Hom}_{\text{sh}}(\Sigma, \text{BC}_2)$. Recall that we are identifying $Q_\Delta = \mathbb{Z}^2$ using the \mathbb{Z} -basis $\Pi_\Delta = \{\alpha_1, \alpha_2\}$ for Q_Δ . So

$$\begin{aligned} s_1 \cdot \chi_{(S,T)}(\lambda) &= s_1(\chi_S(\lambda)\alpha_1 + \chi_T(\lambda)\alpha_2) = \chi_S(\lambda)(-\alpha_1) + \chi_T(\lambda)(2\alpha_1 + \alpha_2) \\ &= (2\chi_T(\lambda) - \chi_S(\lambda), \chi_T(\lambda)). \end{aligned}$$

for $\lambda \in \Sigma$. Also, if $\lambda \in \Pi$, we have $\chi_T(\tilde{w}_{\Pi \setminus T}(\lambda)) = \chi_T(\lambda)$ by (24). Thus

$$\rho_1(\lambda) = 2\chi_T(\lambda) - \chi_S(\tilde{w}_{\Pi \setminus T}(\lambda)) \quad \text{and} \quad \rho_2(\lambda) = \chi_T(\lambda) \quad (31)$$

for $\lambda \in \Pi$.

We check next using (31) that ρ is positive; that is $\rho_1(\lambda) \geq 0$ and $\rho_2(\lambda) \geq 0$ for $\lambda \in \Pi$. First if $\lambda \in T$, we have $\rho_2(\lambda) = 1$. But since $\rho \in \text{Hom}(\Sigma, \text{BC}_2)$, we have $\rho_1(\mu)\rho_2(\mu) \geq 0$ for $\mu \in \Sigma$. So $\rho_1(\lambda) \geq 0$ for $\lambda \in T$. Next, if $\lambda \in \Pi \setminus T$, we have $\rho_2(\lambda) = 0$ and $\rho_1(\lambda) = -\chi_S(\tilde{w}_{\Pi \setminus T}(\lambda)) = \chi_S(\sigma_{\Pi \setminus T}(\lambda)) \geq 0$. So ρ is positive as desired.

Consequently we have $s_1 * \chi_{(S,T)} = \rho$, so $\chi_{s_1 * (S,T)} = \rho$. Thus $s_1 * (S, T) = (\check{S}, \check{T})$, where

$$\check{S} = \{\lambda \in \Pi : \rho_1(\lambda) = 1\} \quad \text{and} \quad \check{T} = \{\lambda \in \Pi : \rho_2(\lambda) = 1\}. \quad (32)$$

It follows now from (31) that $\check{T} = T$, so it remains establish (27).

Now $\check{S} = (\check{S} \setminus T) \cup (\check{S} \cap T)$. Moreover, using (31) and (32),

$$\check{S} \setminus T = \{\lambda \in \Pi \setminus T : \chi_S(\sigma_{\Pi \setminus T}(\lambda)) = 1\} = \sigma_{\Pi \setminus T}(S \setminus T)$$

and

$$\check{S} \cap T = \{\lambda \in T : 2 - \chi_S(\tilde{w}_{\Pi \setminus T}(\lambda)) = 1\} = \{\lambda \in T : \chi_S(\tilde{w}_{\Pi \setminus T}(\lambda)) = 1\}.$$

But if $\lambda \in T$, then

$$\chi_S(\tilde{w}_{\Pi \setminus T}(\lambda)) = \chi_{S \setminus T}(\tilde{w}_{\Pi \setminus T}(\lambda)) + \chi_{S \cap T}(\tilde{w}_{\Pi \setminus T}(\lambda)) = \chi_{S \setminus T}(\tilde{w}_{\Pi \setminus T}(\lambda)) + \chi_{S \cap T}(\lambda)$$

using (24). So we have (27).

Row 3: Let $\tau = (\tau_1, \tau_2) = s_2 \cdot \chi_{(S,T)} \cdot \tilde{w}_{\Pi \setminus S} \in \text{Hom}_{\text{sh}}(\Sigma, \text{BC}_2)$. Now

$$\begin{aligned} s_2 \cdot \chi_{(S,T)}(\lambda) &= s_2(\chi_S(\lambda)\alpha_1 + \chi_T(\lambda)\alpha_2) = \chi_S(\lambda)(\alpha_1 + \alpha_2) + \chi_T(\lambda)(-\alpha_2) \\ &= (\chi_S(\lambda), \chi_S(\lambda) - \chi_T(\lambda)). \end{aligned}$$

for $\lambda \in \Sigma$. Also if $\lambda \in \Pi$, $\chi_S(\tilde{w}_{\Pi \setminus S}(\lambda)) = \chi_S(\lambda)$ by (24). Thus

$$\tau_1(\lambda) = \chi_S(\lambda) \quad \text{and} \quad \tau_2(\lambda) = \chi_S(\lambda) - \chi_T(\tilde{w}_{\Pi \setminus S}(\lambda)) \quad (33)$$

for $\lambda \in \Pi$. Arguing as in Row 2 we now easily see that τ is positive, $\chi_{s_2 * (S,T)} = \tau$ and $s_2 * (S, T) = (\bar{S}, \bar{T})$, where

$$\bar{S} = \{\lambda \in \Pi : \tau_1(\lambda) = 1\} \quad \text{and} \quad \bar{T} = \{\lambda \in \Pi : \tau_2(\lambda) = 1\}.$$

It follows then from (33) that $\bar{S} = S$, so it remains to prove (28). Again arguing as in Row 2, we easily see that

$$\bar{T} = \sigma_{\Pi \setminus S}(T \setminus S) \cup \{\lambda \in S : \chi_{T \setminus S}(\tilde{w}_{\Pi \setminus S}(\lambda)) + \chi_{T \cap S}(\lambda) = 0\}.$$

Finally, let $\lambda \in S$. Then, since $\tilde{w}_{\Pi \setminus S}(\lambda) \in \lambda + Q_{\Pi \setminus S}$, we see that $\tilde{w}_{\Pi \setminus S}(\lambda)$ is positive and hence $\chi_{T \setminus S}(\tilde{w}_{\Pi \setminus S}(\lambda)) \geq 0$. Thus $\chi_{T \setminus S}(\tilde{w}_{\Pi \setminus S}(\lambda)) + \chi_{T \cap S}(\lambda) = 0$ if and only if $\lambda \in S \setminus T$ and $\chi_{T \setminus S}(\tilde{w}_{\Pi \setminus S}(\lambda)) = 0$, which holds if and only if $\lambda \in S \setminus T$ and $\text{supp}_{\Pi}(\tilde{w}_{\Pi \setminus S}(\lambda)) \cap (T \setminus S) = \emptyset$.

Row 4: Using Row 2 and Row 3 (applied to (\check{S}, T)), we have

$$(s_2 s_1) * (S, T) = s_2 * (s_1 * (S, T)) = s_2 * (\check{S}, T) = (\check{S}, T')$$

for some subset T' of Π . On the other hand, $s_2 s_1 = -s_1 s_2$. So, using Row 3, Row 2 (applied to (S, \bar{T})) and Row 5 (applied to (S', \bar{T})), we have

$$\begin{aligned} (s_2 s_1) * (S, T) &= (-1) * (s_1 * (s_2 * (S, T))) \\ &= (-1) * (s_1 * (S, \bar{T})) = (-1) * (S', \bar{T}) = (S' \cdot \sigma_{\Pi}, \bar{T} \cdot \sigma_{\Pi}) \end{aligned}$$

for some subset S' of Π . Combining these equalities gives our conclusion. \square

If $(S, T) \in \text{SPA}(\Pi)$, then we see using Theorems 8.3.1 and 8.3.2 that

$$\mathfrak{P}(S, T)^{\text{op}} = {}^{-1}\mathfrak{P}(S, T) \simeq_{\text{gr}} \mathfrak{P}((-1) * (S, T)) = \mathfrak{P}(S \cdot \sigma_{\Pi}, T \cdot \sigma_{\Pi}) \simeq_{\text{gr}} \mathfrak{P}(S, T).$$

This together with Theorem 7.2.4(ii) gives another proof of Proposition 6.2.2.

The following corollary, which we state for emphasis and convenience of reference, follows taking $u = s_1$ and $u = s_2$ in Theorems 8.3.1 and 8.3.2.

Corollary 8.3.3. *If $(S, T) \in \text{SPA}(\Pi)$ and \check{S} and \bar{T} are given by (29) and (30) (or (27) and (28)), then*

$$\mathfrak{P}(S, T)^{\circ} \simeq_{\text{gr}} \mathfrak{P}(\check{S}, T) \quad \text{and} \quad \overline{\mathfrak{P}(S, T)} \simeq_{\text{gr}} \mathfrak{P}(S, \bar{T}).$$

Remark 8.3.4. Corollary 8.3.3 explains our choice of notation for \check{S} and \bar{T} . It should be noted however that this is a (convenient) abuse of notation, since \check{S} and \bar{T} each depend on *both* S and T .

Example 8.3.5 (The trivial SP-gradings). Suppose that $S \in \text{KA}(\Pi)$. Recall from Section 7.2 that the zero and one SP-gradings on $\mathfrak{P}(S)$ are determined by \emptyset and S respectively. For these SP-gradings one can check easily using Corollary 8.3.3, (29) and (30) that

$$\mathfrak{P}(S, \emptyset)^{\circ} \simeq_{\text{gr}} \mathfrak{P}(S, \emptyset), \quad \mathfrak{P}(S, S)^{\circ} \simeq_{\text{gr}} \mathfrak{P}(S, S),$$

and

$$\overline{\mathfrak{P}(S, \emptyset)} \simeq_{\text{gr}} \mathfrak{P}(S, S), \quad \overline{\mathfrak{P}(S, S)} \simeq_{\text{gr}} \mathfrak{P}(S, \emptyset).$$

The computation of Weyl images is particularly simple in the close-to-Jordan case.

Corollary 8.3.6. *Suppose that $(S, T) \in \text{SPA}(\Pi)$ and $P = \mathfrak{P}(S, T)$ is close-to-Jordan with non-trivial SP-grading. Then*

$$\mathfrak{P}(S, T)^{\circ} \simeq_{\text{gr}} \mathfrak{P}(\sigma_{\Pi \setminus T}(S \setminus T), T) \quad \text{and} \quad \overline{\mathfrak{P}(S, T)} \simeq_{\text{gr}} \mathfrak{P}(S, T).$$

Proof. Recall that $X_n \neq A_1$, S is the set of nodes of Π that are adjacent to μ^- in $\tilde{\Pi}$, and $T = \{\lambda\}$ with $\lambda \in \Pi$ and $\chi_\lambda(\mu^+) = 1$ (see Theorems 5.4.1(ii) and 7.3.1). We assume that λ is not an end node of Π if $X_n = A_n$, leaving the excluded case for the reader to check. Now $\chi_S(\mu^+) = 2$ by Theorem 5.4.1(i). Also $S \cap T = \emptyset$. Indeed this holds by assumption if $X_n = A_n$, whereas it holds when $X_n \neq A_n$ since in that case $\text{card}(S) = 1$ by Remark 5.4.2(i). Thus, we see from Corollary 8.3.3, (29) and (30), that $\mathfrak{P}(S, T)^\sim \simeq_{\text{gr}} \mathfrak{P}(\sigma_{\Pi \setminus T}(S \setminus T), T)$ and $\overline{\mathfrak{P}(S, T)} \simeq_{\text{gr}} \mathfrak{P}(S, \sigma_{\Pi \setminus S}(T \setminus S))$. Finally, by uniqueness in Corollary 7.3.2, we have $\overline{\mathfrak{P}(S, T)} \simeq_{\text{gr}} \mathfrak{P}(S, T)$ if $X_n \neq A_n$ and $X_n \neq D_n$. But if $X_n = A_n$ or $X_n = D_n$, $\sigma_{\Pi \setminus S}$ extends to an automorphism of Π , so $\overline{\mathfrak{P}(S, T)} \simeq_{\text{gr}} \mathfrak{P}(S, T)$ by Theorem 7.3.1. \square

Example 8.3.7. (Type E_6). Recall that in Example 7.4.1 we saw that there are, up to graded-isomorphism, seven simple SP-graded Kantor pairs of type E_6 whose gradings are non-trivial. It is straightforward to apply Corollary 8.3.3, together with (29) and (30), to calculate the reflection and shift of each of these SP-graded Kantor pairs up to graded-isomorphism. One sees that

shifting exchanges $E_6(20, 5, 8)$ and $E_6(20, 5, 12)$;

and that shifting fixes the other five graded Kantor pairs. One also sees that

reflection exchanges $E_6(16, 0, 8)$ and $E_6(16, 8, 8a)$;

reflection exchanges $E_6(20, 1, 10)$ and $E_6(20, 5, 10)$;

and that reflection fixes $E_6(20, 5, 8)$, $E_6(20, 5, 12)$ and $E_6(16, 8, 8b)$. In particular, reflection does not preserve balanced 2-dimension. In fact, we see that the reflection of the Jordan pair $E_6(16, 0, 8)$ is not Jordan, and that the reflection of the close-to-Jordan pair $E_6(20, 1, 10)$ is not close-to-Jordan.

8.4. The proofs of (29) and (30). We now return to the proofs of (29) and (30) that we postponed earlier. The reader may elect to further postpone reading these arguments, as they will not be used in the final section.

The information that we need about the Weyl group elements $\tilde{w}_{\Pi \setminus T}$ and $\tilde{w}_{\Pi \setminus S}$ is provided by the next two lemmas:

Lemma 8.4.1. *If $(S, T) \in \text{SPA}(\Pi)$, then $\chi_{S \setminus T}(\tilde{w}_{\Pi \setminus T}(\lambda)) = \chi_{S \setminus T}(\mu^+)$ for $\lambda \in T$.*

Proof. Let $\lambda \in T$. We can assume

$$\nu_0 := \tilde{w}_{\Pi \setminus T}(\lambda) \neq \mu^+.$$

Now $\chi_T(\tilde{w}_{\Pi \setminus T}(\lambda)) = \chi_T(\lambda)$ by (24), so

$$\chi_T(\nu_0) = 1.$$

Hence $\nu_0 \in \Sigma^+$, so there are $\lambda_1, \dots, \lambda_r$ in Π with $r \geq 1$ such that

$$\nu_i := \nu_{i-1} + \lambda_i \in \Sigma^+ \text{ for } 1 \leq i \leq r \text{ and } \nu_r = \mu^+.$$

We next claim that $\lambda_1 \in T$. Indeed, otherwise $\lambda_1 \in \Pi \setminus T = \sigma_{\Pi \setminus T}(\Pi \setminus T)$, so $\lambda_1 = \sigma_{\Pi \setminus T}(\nu)$ for some $\nu \in \Pi \setminus T$. Hence

$$\tilde{w}_{\Pi \setminus T}(\lambda - \nu) = \tilde{w}_{\Pi \setminus T}(\lambda) - \tilde{w}_{\Pi \setminus T}(\nu) = \nu_0 + \lambda_1 = \nu_1 \in \Sigma$$

and therefore $\lambda - \nu \in \Sigma$. This is a contradiction since $\lambda \in T$ and $\nu \in \Pi \setminus T$. So we have our claim.

Next, for $1 \leq p \leq r$, we have

$$\chi_T(\nu_p) = \chi_T(\nu_0 + \lambda_1 + \dots + \lambda_p) \geq \chi_T(\nu_0) + \chi_T(\lambda_1) = 1 + 1 = 2.$$

Hence $\chi_T(\nu_p) = 2$ for $1 \leq p \leq r$. Thus, since $(S, T) \in \text{SPA}(\Pi)$, we have $\chi_S(\nu_p) \neq 1$ for $1 \leq p \leq r$. So, for $2 \leq p \leq r$, we have $\chi_S(\lambda_p) = \chi_S(\nu_p) - \chi_S(\nu_{p-1}) \in 2\mathbb{Z}$ and hence $\lambda_p \notin S$.

It follows from the previous two paragraphs that $\lambda_p \notin S \setminus T$ for $1 \leq p \leq r$. So $\chi_{S \setminus T}(\nu_0) = \chi_{S \setminus T}(\nu_r) = \chi_{S \setminus T}(\mu^+)$. \square

Lemma 8.4.2. *If $X \subseteq \Pi$ and $\lambda \in \Pi \setminus X$, then $\text{supp}_\Pi(\tilde{w}_X(\lambda)) = \text{comp}(X \cup \{\lambda\}, \lambda)$.*

Proof. Let Y_1, \dots, Y_r be the connected components of $X \cup \{\lambda\}$ with $\lambda \in Y_1$, and let Z_1, \dots, Z_s be the connected components of $Y_1 \setminus \{\lambda\}$. Then $Z_1, \dots, Z_s, Y_2, \dots, Y_r$ are the connected components of X . We must show that $V = Y_1$, where $V = \text{supp}_\Pi(\tilde{w}_X(\lambda))$.

Now it is clear that $\tilde{w}_X = \tilde{w}_{Z_1} \dots \tilde{w}_{Z_r} \tilde{w}_{Y_2} \dots \tilde{w}_{Y_r}$, so $\tilde{w}_X(\lambda) = \tilde{w}_{Z_1} \dots \tilde{w}_{Z_r}(\lambda) \in \lambda + Q_{Z_1 \cup \dots \cup Z_s} \subseteq Q_{Y_1}$. Thus $\lambda \in V \subseteq Y_1$.

Next, to show that $Y_1 \subseteq V$, it suffices to show that any $\mu \in X \cup \{\lambda\}$ that is adjacent to some $\nu \in V$ lies in V . For this we can assume that $\mu \neq \lambda$, so $\mu \in X$. Then $\mu = \sigma_X(\mu')$ for some $\mu' \in X$, so $\tilde{w}_X(\lambda) + \mu = \tilde{w}_X(\lambda - \mu') \notin \Sigma$. Hence $\langle \tilde{w}_X(\lambda), \mu \rangle \geq 0$. Since $\langle \nu, \mu \rangle < 0$, this forces $\langle \nu', \mu \rangle > 0$ for some $\nu' \in V$. So, since $\mu, \nu' \in \Pi$, we have $\mu = \nu'$, and hence $\mu \in V$. \square

Proof of (29). By (27) and Lemma 8.4.1, $\check{S} = \sigma_{\Pi \setminus T}(S \setminus T) \cup (\check{S} \cap T)$ and

$$\check{S} \cap T = \{\lambda \in T : \chi_{S \setminus T}(\mu^+) + \chi_{S \cap T}(\lambda) = 1\}$$

Furthermore, by Proposition 7.1.5(c), $\chi_S(\mu^+) \in \{1, 2\}$, so $\chi_{S \setminus T}(\mu^+) \in \{0, 1, 2\}$. If $\chi_{S \setminus T}(\mu^+) = 0$ (or equivalently $S \subseteq T$), then $\check{S} \cap T = \{\lambda \in T : \chi_S(\lambda) = 1\} = S$. Also, if $\chi_{S \setminus T}(\mu^+) = 1$, then $\check{S} \cap T = \{\lambda \in T : \chi_{S \cap T}(\lambda) = 0\} = T \setminus S$. Finally, if $\chi_{S \setminus T}(\mu^+) = 2$, then $\check{S} \cap T = \{\lambda \in T : \chi_{S \cap T}(\lambda) = -1\} = \emptyset$. \square

Proof of (30). By (28) and Lemma 8.4.2 (with $X = \Pi \setminus S$), we have

$$\bar{T} = \sigma_{\Pi \setminus S}(T \setminus S) \bigcup \{\lambda \in S \setminus T : \text{comp}((\Pi \setminus S) \cup \{\lambda\}, \lambda) \cap (T \setminus S) = \emptyset\}. \quad (34)$$

If $T \subseteq S$, then $T \setminus S = \emptyset$, so $\bar{T} = S \setminus T$ by (34). Next if (ex) holds, then it is easy to check using (34) that $\bar{T} = \{\sigma_{\Pi \setminus S}(\mu), \lambda\}$. We leave this to the reader.

Finally suppose that $T \not\subseteq S$ and (ex) does not hold. We suppose for contradiction (which will complete the proof of (30)) that

$$\text{comp}((\Pi \setminus S) \cup \{\lambda\}, \lambda) \cap (T \setminus S) = \emptyset, \quad (35)$$

for some $\lambda \in S \setminus T$. If $S = \{\lambda\}$, then $(\Pi \setminus S) \cup \{\lambda\} = \Pi$ which is connected, so, by (35), we have $T \setminus S = \emptyset$, giving a contradiction. Hence we can assume that S contains an element $\lambda' \neq \lambda$. Thus, by Proposition 7.1.5(c), we have $S = \{\lambda, \lambda'\}$, with

$$\chi_\lambda(\mu^+) = \chi_{\lambda'}(\mu^+) = 1. \quad (36)$$

Then $(\Pi \setminus S) \cup \{\lambda\} = \Pi \setminus \{\lambda'\}$ and $T \setminus S = T \setminus \{\lambda'\}$ since $\lambda \notin T$. So, by (35), we have

$$\text{comp}(\Pi \setminus \{\lambda'\}, \lambda) \cap T = \emptyset. \quad (37)$$

Now if $\Pi \setminus \{\lambda'\}$ is connected, then $(\Pi \setminus \{\lambda'\}) \cap T = \emptyset$ implies that $T \subseteq \{\lambda'\} \subseteq S$, a contradiction. So

$$\Pi \setminus \{\lambda'\} \text{ has at least 2 connected components.} \quad (38)$$

Now a check of μ^+ for each type shows that the existence of distinct elements $\lambda, \lambda' \in \Pi$ satisfying both (36) and (38) implies that Π has type A_n , where $n \geq 3$. Then, by (37), $\text{comp}(\Pi \setminus \{\lambda'\}, \lambda) \cup \{\mu^-\}$ is a connected subset of $\tilde{\Pi} \setminus T$, so

$$\lambda \in \text{comp}(\tilde{\Pi} \setminus T, \mu^-) \cap S.$$

Therefore, by Proposition 7.1.5(c), we have $\chi_T(\mu^+) = 0$ or 1. But certainly $\chi_T(\mu^+) \neq 0$ since $T \neq \emptyset$. Hence $\chi_T(\mu^+) = 1$, so $T = \{\mu\}$ for some $\mu \in \Pi$, with $\mu \neq \lambda'$ since $T \not\subseteq S$. It is now clear from (37) that we have (ex), giving a contradiction. \square

9. REFLECTIONS OF SIMPLE CLOSE-TO-JORDAN PAIRS

We continue with the assumptions and notation of Section 7. In this section we use results from Sections 5–8 to give a construction of the reflection of each simple SP-graded close-to-Jordan pair of type X_n whose grading is non-trivial. We do this using the description of these graded pairs given in Theorem 7.3.1.

9.1. The trilinear pair $\mathfrak{T}(J, \tau)$. In order to construct some trilinear pairs, we fix a pair $U = (U^-, U^+)$ of 2-dimensional vector spaces and a nondegenerate bilinear map $(\cdot, \cdot) : U^- \times U^+ \rightarrow \mathbb{K}$; and we set $(u^+, u^-) = (u^-, u^+)$ for $u^\sigma \in U^\sigma$.

In order to construct gradings on our trilinear pairs, we fix bases $\{u_0^\sigma, u_1^\sigma\}$ for U^σ , $\sigma = \pm$, such that $(u_i^-, u_j^+) = \delta_{ij}$ (so these bases are dual with respect to (\cdot, \cdot)).

Construction 9.1.1. (The graded trilinear pair $\mathfrak{T}(J, \tau)$) Let J be a trilinear pair with products $\{\cdot, \cdot, \cdot\}_J^\sigma$, $\sigma = \pm$, and define $D_J^\sigma(x, a) \in \text{End}(J^\sigma)$ by $D_J^\sigma(x, a)y = \{x, a, y\}_J^\sigma$. Also, let $\tau = (\tau^-, \tau^+)$, where $\tau^\sigma : J^\sigma \times J^{-\sigma} \rightarrow \mathbb{K}$ is a bilinear map for $\sigma = \pm$. Then

$$\mathfrak{T}(J, \tau) = J \otimes U := (J^- \otimes U^-, J^+ \otimes U^+)$$

is a trilinear pair (but not in general a Kantor pair) with products $\{\cdot, \cdot, \cdot\}^\sigma$ given by

$$\{x \otimes r, a \otimes \ell, y \otimes s\}^\sigma = \{x, a, y\}_J^\sigma \otimes (r, \ell)s - \tau^\sigma(x, a)y \otimes (r, \ell, s) \quad (39)$$

for $x, y \in J^\sigma$, $a \in J^{-\sigma}$, $r, s \in U^\sigma$, $\ell \in U^{-\sigma}$, where

$$(r, \ell, s) = (r, \ell)s - (s, \ell)r$$

for $r, s \in U^\sigma$, $\ell \in U^{-\sigma}$. Moreover, one checks easily that $\mathfrak{T}(J, \tau) = \bigoplus_{i \in \mathbb{Z}} \mathfrak{T}(J, \tau)_i = \mathfrak{T}(J, \tau)_0 \oplus \mathfrak{T}(J, \tau)_1$ is a \mathbb{Z} -graded trilinear pair with

$$\mathfrak{T}(J, \tau)_i = J^\sigma \otimes u_i^\sigma$$

for $i = 0, 1$ and $\mathfrak{T}(J, \tau)_i = 0$ otherwise.

It is clear that the trilinear pair $\mathfrak{T}(J, \tau)$ does not depend up to isomorphism on the choice of U^-, U^+ or (\cdot, \cdot) , and that the grading $\mathfrak{T}(J, \tau) = \mathfrak{T}(J, \tau)_0 \oplus \mathfrak{T}(J, \tau)_1$ does not depend up to isomorphism on the choice of the dual bases $\{u_0^\sigma, u_1^\sigma\}$.

Remark 9.1.2. The above construction of trilinear pairs is a pair version, with basis free products, of a construction of triple systems given by Kantor in [K1]. More precisely suppose that $J = (X, X)$ is the double of a triple system X and $\tau^- = \tau^+ : X \times X \rightarrow \mathbb{K}$ is a bilinear map. Then one can easily check that the pair $\mathfrak{T}(J, \tau)$ is isomorphic to the double of the Kronecker product $X \otimes \mathbb{K}^2$ defined in [K1, §6. Defn. 6] with $k = 2$, $\lambda = \mu = 1$ and $f(x, y) = \tau^\sigma(y, x)$, $\sigma = \pm$. (It appears however that there is a typo in Kantor's definition: the defining expression for $(YXZ)_i$ should read as $\sum_{\alpha=1}^k ((y_\alpha x_\alpha z_i) + \lambda f(x_\alpha, y_i)z_\alpha - \mu f(x_\alpha, y_\alpha)z_i)$.)

9.2. Constructing the reflection of $\mathfrak{P}(\Pi; S, T)$. Suppose for the rest of the article that $(S, T) \in \text{SPA}(\Pi)$ and that $\mathfrak{P}(\Pi; S, T)$ is close-to Jordan with nontrivial SP-grading. Our goal is to construct the reflection of $\mathfrak{P}(\Pi; S, T)$ in the form $\mathfrak{T}(J, \tau)$ for a specified choice of J and τ .

As noted in Corollary 7.3.2, our assumptions imply that $X_n = A_n$ ($n \geq 2$), B_n ($n \geq 2$), C_n ($n \geq 3$), D_n ($n \geq 4$), E_6 or E_7 . In particular, $n \geq 2$.

We write the distinct elements of Π as μ_1, \dots, μ_n . By Theorems 5.4.1 and 7.3.1, we know that S is the set of nodes of Π that are adjacent to μ^- in $\tilde{\Pi}$, $\chi_S(\mu^+) = 2$ and

$$T = \{\mu_t\} \text{ for some } 1 \leq t \leq n \text{ with } \chi_{\mu_t}(\mu^+) = 1.$$

We let

$$\Pi' = \Pi \setminus T \quad \text{and} \quad S' = S \setminus T.$$

In the remainder of this section we will for convenience exclude from consideration the case when $X_n = A_n$ and μ_t is an interior (not an end) node of Π . We will return to consider this excluded case in the last subsection.

Lemma 9.2.1. *If $X_n = A_n$, suppose that the element μ_t of T is an end node of Π . Then*

$$S = \begin{cases} \{\mu_s, \mu_t\} \text{ where } \mu_s \text{ is the other end node in } \Pi, & \text{if } X_n = A_n; \\ \{\mu_s\} \text{ where } \mu_s \in \Pi \text{ with } \chi_{\mu_s}(\mu^+) = 2, & \text{otherwise.} \end{cases}$$

Also there exists a unique $\mu_{t'}$ in Π' that is adjacent to μ_t in Π , so Π' is connected.

Proof. The first statement follows from Remark 5.4.2(i), and the second is easily checked considering types case-by-case. \square

With the assumptions and notation of Lemma 9.2.1, we define a constant $\theta_{\Pi, T} \in \mathbb{Q}$ by

$$\theta_{\Pi, T} := \begin{cases} p + 1 & \text{if } X_n = A_n \text{ (} n \geq 2 \text{)} \\ p + 2 & \text{otherwise,} \end{cases}$$

where p is the product of the (t, t') -entry of the Cartan matrix $C(\Pi)$ and the (t', s) -entry of the inverse of $C(\Pi')$. (Note that $C(\Pi)$ is an $I \times I$ matrix, where $I = \{1, \dots, n\}$, whereas $C(\Pi')$ is an $I' \times I'$ matrix with $I' = I \setminus \{t\}$. See Subsection 1.1.)

Theorem 9.2.2. *Suppose that $(S, T) \in \text{SPA}(\Pi)$, $P = \mathfrak{P}(\Pi; S, T)$ is close-to Jordan with nontrivial SP-grading, and, if $X_n = A_n$, the element μ_t of T is an end node of Π . Then*

- (i) P_0 is a simple Jordan pair that is isomorphic to $\mathfrak{P}(\Pi'; S')$.
- (ii) Suppose that J is any simple Jordan pair that is isomorphic to $\mathfrak{P}(\Pi'; S')$, and let $\tau = \tau_{J, \Pi, T} := (\tau^-, \tau^+)$, where $\tau^\sigma : J^\sigma \times J^{-\sigma} \rightarrow \mathbb{K}$ is given by

$$\tau^\sigma(x, a) = \frac{\theta_{\Pi, T}}{\dim(J^\sigma)} \text{tr}(D_J^\sigma(x, a)) \quad (40)$$

for $x \in J^\sigma$, $a \in J^{-\sigma}$, $\sigma = \pm$. Then $\tau^\sigma(x, a) = \tau^{-\sigma}(a, x)$; τ^σ is non-degenerate for $\sigma = \pm$; $\mathfrak{T}(J, \tau)$ is a simple SP-graded Kantor pair; and

$$\check{P} \simeq_{gr} \mathfrak{T}(J, \tau).$$

Proof. We first set some notation. As usual, choose $h_i \in [\mathcal{G}_{\mu_i}, \mathcal{G}_{-\mu_i}]$ so that $\mu_i(h_i) = 2$ for $1 \leq i \leq n$, in which case $\{h_i\}_{i=1}^n$ is a basis for \mathcal{H} .

Also, choose nonzero $e^\sigma \in \mathcal{G}_{\mu^\sigma}$ for $\sigma = \pm$ such that $[h^+, e^\sigma] = \sigma 2e^\sigma$, where $h^+ = [e^+, e^-]$. Then since $\chi_S(\mu^+) = 2$ and $\chi_T(\mu^+) = 1$, we see by (12) that

$$\mathcal{G}(\chi_{(S,T)})_{\sigma 2, *} = \mathcal{G}(\chi_{(S,T)})_{\sigma 2, \sigma 1} = \mathcal{G}_{\mu^\sigma} = \mathbb{K}e^\sigma. \quad (41)$$

(i): Let $\Sigma' = \{\mu \in \Sigma : \chi_T(\mu) = 0\}$. Then, since Π' is connected, Σ' is an irreducible root system of rank $n-1$ (in its real span) with base Π' . Let \mathcal{G}' be the subalgebra of \mathcal{G} generated by $\{\mathcal{G}_{\mu_i} + \mathcal{G}_{-\mu_i} : i \neq t\}$, and let $\mathcal{H}' = \mathcal{H} \cap \mathcal{G}'$. Then

$$\mathcal{G}' = \mathcal{H}' \oplus \left(\bigoplus_{\mu \in \Sigma'} \mathcal{G}_\mu \right); \quad \mathcal{H}' = \sum_{i \neq t} \mathbb{K}h_i;$$

\mathcal{G}' is a simple Lie algebra with Cartan subalgebra \mathcal{H}' ; and $\Sigma(\mathcal{G}', \mathcal{H}') = \Sigma'$ (identifying elements of Σ' with their restrictions to \mathcal{H}').

We now use \mathcal{G}' , \mathcal{H}' , Π' and $S' \in \text{KA}(\Pi')$ in Construction 5.2.1 to obtain a 5-graded Lie algebra $\mathcal{G}'(\chi_{S'}) = \bigoplus_{i \in \mathbb{Z}} \mathcal{G}'(\chi_{S'})_i$ and hence the Kantor pair $\mathfrak{P}(\Pi'; S')$ enveloped by $\mathcal{G}'(\chi_{S'})$ (see Construction 5.2.1). Note that for $i \neq 0$ we have

$$\mathcal{G}'(\chi_{S'})_i = \sum_{\mu \in \Sigma', \chi_{S'}(\mu)=i} \mathcal{G}'_\mu = \sum_{\mu \in \Sigma, \chi_S(\mu)=i, \chi_T(\mu)=0} \mathcal{G}_\mu = \mathcal{G}(\chi_{(S,T)})_{i,0}.$$

Hence

$$\mathfrak{P}(\Pi'; S')^\sigma = \mathcal{G}'(\chi_{S'})_{\sigma 1} = \mathcal{G}(\chi_{(S,T)})_{\sigma 1,0} = P_0^\sigma, \quad (42)$$

so $P_0 = \mathfrak{P}(\Pi'; S')$. Moreover, $\mathcal{G}'(\chi_{S'})_{\sigma 2} = \mathcal{G}(\chi_{(S,T)})_{\sigma 2,0} = 0$ by (41), so $\mathcal{G}'(\chi_{S'})$ is 3-graded and thus P_0 is Jordan by Remark 5.2.3(ii).

(ii): To prove (ii), we can assume that $J = P_0$ and use the notation and conclusions in the proof of (i).

Let $\omega := \exp(\text{ad } e^+) \exp(-\text{ad } e^-) \exp(\text{ad } e^+) \in \text{Aut}(\mathcal{G})$. We will use Lemma 5.5.1, which gives us detailed information about ω .

First, by Lemma 5.5.1(v), $\omega(x) = -\sigma[e^{-\sigma}, x]$ for $x \in P^\sigma$. Thus, since $e^{-\sigma} \in \mathcal{G}(\chi_{(S,T)})_{-\sigma 2, -\sigma 1}$ by (41), we see that $\omega(\mathcal{G}(\chi_{(S,T)})_{\sigma 1, \sigma i}) = \mathcal{G}(\chi_{(S,T)})_{-\sigma 1, -\sigma(1-i)}$ for $i = 0, 1$. That is

$$\omega \text{ exchanges } P_i^\sigma \text{ and } P_{1-i}^{-\sigma}.$$

Next let $\tau^\sigma = \zeta^\sigma|_{J^\sigma \times J^{-\sigma}} : J^\sigma \times J^{-\sigma} \rightarrow \mathbb{K}$, with ζ^σ as defined in Lemma 5.5.1(vi), so

$$[[x, a], e^\sigma] = \tau^\sigma(x, a)e^\sigma \quad (43)$$

for $x \in J^\sigma$, $a \in J^{-\sigma}$. At this point we will take this as the definition of τ^σ , and then at the end of the proof we will prove (40).

Note that $\tau^\sigma(x, a) = \tau^{-\sigma}(a, x)$ by (16). Also, for $\sigma = \pm$, $i = 0, 1$, we have

$$[[P_i^\sigma, P_{1-i}^{-\sigma}], e^\sigma] \subseteq [[\mathcal{G}(\chi_{(S,T)})_{\sigma 1, \sigma i}, \mathcal{G}(\chi_{(S,T)})_{-\sigma 1, -\sigma(1-i)}], \mathcal{G}(\chi_{(S,T)})_{\sigma 2, \sigma 1}],$$

which is contained in $\mathcal{G}(\chi_{(S,T)})_{\sigma 2, \sigma 2i} = 0$. Thus, by (14), $\zeta^\sigma(P_i^\sigma, P_{1-i}^{-\sigma}) = 0$. So the non-degeneracy of ζ^σ implies that of τ^σ .

In the rest of the proof, it is more convenient to work with $Q := \check{P}^{\text{op}}$, rather than \check{P} itself. We next show that the \mathbb{Z} -graded trilinear pairs Q and $\mathfrak{T}(J, \tau)$ are graded-isomorphic. This will show that $\mathfrak{T}(J, \tau)$ is a simple SP-graded Kantor pair and, by Proposition 6.2.2, that $\check{P} \simeq_{\text{gr}} \mathfrak{T}(J, \tau)$ (leaving only (40) to prove).

Now $Q = Q^0 \oplus Q^1$, where $Q_0^\sigma = P_0^{-\sigma} = P_0^\sigma$ and $Q_1^\sigma = P_1^{-\sigma} = P_1^\sigma = \omega(P_0^\sigma)$, so

$$Q_i^\sigma = \omega^i(J^\sigma)$$

for $\sigma = \pm$, $i = 0, 1$. Moreover, by the definition of Weyl images (see Section 6.3), the products in Q are given by $\{x, a, y\}^\sigma = [[x, a], y]$ in \mathcal{G} . On the other hand,

$\mathfrak{T}(J, \tau) = \mathfrak{T}(J, \tau)_0 \oplus \mathfrak{T}(J, \tau)_1$ with $\mathfrak{T}(J, \tau)_i = J^\sigma \otimes u_i^\sigma$ and products given by (39). With this in mind, we define $\varphi = (\varphi^-, \varphi^+)$, where $\varphi^\sigma : \mathfrak{T}(J, \tau)^\sigma \rightarrow Q^\sigma$ is the linear isomorphism such that

$$\varphi^\sigma(x \otimes u_i^\sigma) = \omega^i x$$

for $x \in J^\sigma$, $\sigma = \pm$ and $i = 0, 1$. In order to prove that φ is an isomorphism of trilinear pairs, we must show that

$$\varphi^\sigma(\{x \otimes u_i^\sigma, a \otimes u_j^{-\sigma}, y \otimes u_k^\sigma\}) = [[\omega^i x, \omega^j a], \omega^k y]$$

for $\sigma = \pm$, $x, y \in J^\sigma$, $a \in J^{-\sigma}$ and $i, j, k = 0, 1$. But $(u_i^\sigma, u_j^{-\sigma})u_k^\sigma = \delta_{ij}u_k^\sigma$ and $(u_i^\sigma, u_j^{-\sigma}, u_k^\sigma) = \delta_{ij}u_k^\sigma - \delta_{jk}u_i^\sigma$. So we must prove that

$$\delta_{ij}\omega^k([x, a], y) - \delta_{ij}\tau^\sigma(x, a)\omega^k y + \delta_{jk}\tau^\sigma(x, a)\omega^i y = [[\omega^i x, \omega^j a], \omega^k y]. \quad (44)$$

Now if $(i, j, k) = (0, 0, 0)$, (44) is trivial; whereas if $(i, j, k) = (1, 1, 1)$, (44) holds since ω is an automorphism. If $(i, j, k) = (1, 0, 0)$ or $(0, 0, 1)$, then (44) follows from Lemma 5.5.1(vii); whereas if $(i, j, k) = (0, 1, 0)$, (44) holds since its right hand side lies in $\mathcal{G}(\chi_{(S,T)})\sigma 3, *$, which is 0. Finally the cases $(1, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$ follow by applying ω to the cases $(0, 0, 1)$, $(0, 1, 0)$ and $(1, 0, 0)$ respectively.

It remains to prove (40). For this let $\{\ell'_i\}_{i \neq t}$ be the basis for \mathcal{H}' that is dual to $\{\mu_i\}_{i \neq t}$.

Recall from the proof of (i) that $\mathcal{G}'(\chi_{S'}) = \bigoplus_{i \in \mathbb{Z}} \mathcal{G}'(\chi_{S'})_i$ is 3-graded and that $\mathcal{G}'(\chi_{S'})_{\sigma 1} = P_0^\sigma = J^\sigma$. To simplify notation, we set $\mathfrak{f} := \mathcal{G}'(\chi_{S'})_0$. Then

$$\mathfrak{f} = \mathcal{H}' \oplus \sum_{\mu \in \Sigma', \chi_{S'}(\mu)=0} \mathcal{G}'_\mu = \mathcal{H}' \oplus \sum_{\mu \in \Sigma, \chi_S(\mu)=0, \chi_T(\mu)=0} \mathcal{G}_\mu = \mathbb{K}h_s \oplus \mathfrak{k} \quad (45)$$

as vector spaces, where \mathfrak{k} is the subalgebra of \mathcal{G} generated by $\{\mathcal{G}_{\mu_i} + \mathcal{G}_{-\mu_i} : i \neq s, t\}$. Further, \mathfrak{k} is semi-simple and ℓ'_s is in the centre of \mathfrak{f} . So $\ell'_s \notin \mathfrak{k}$ and hence

$$\mathfrak{f} = \mathbb{K}\ell'_s \oplus \mathfrak{k}. \quad (46)$$

as algebras. We next construct some elements of the one-dimensional space

$$\mathcal{F} := \{\lambda \in \text{Hom}(\mathfrak{f}, \mathbb{K}) : \lambda(\mathfrak{k}) = 0\}.$$

First note that $\mathfrak{f} \subseteq \mathcal{G}(\chi_S)_{0,0}$ by (45). So by (41), $[\mathfrak{f}, e^\sigma] \subseteq \mathbb{K}e^\sigma$ for $\sigma = \pm$. Thus for $\sigma = \pm$, there exists a unique $\nu^\sigma \in \text{Hom}(\mathfrak{f}, \mathbb{K})$ such that

$$[z, e^\sigma] = \nu^\sigma(z)e^\sigma$$

for $z \in \mathfrak{f}$. Then since $\mathfrak{k} = [\mathfrak{k}, \mathfrak{k}]$, we have $\nu^\sigma \in \mathcal{F}$. Also $\nu^\sigma(\ell'_s) = \mu^\sigma(\ell'_s) = \sigma\mu^+(\ell'_s)$.

Next, since $\mathcal{G}'(\chi_{S'})$ is 3-graded, we have $[\mathfrak{f}, J^\sigma] \subseteq J^\sigma$ for $\sigma = \pm$. So we can define $\xi^\sigma \in \text{Hom}(\mathfrak{f}, \mathbb{K})$ by

$$\xi^\sigma(z) = \text{tr}(\text{ad}(z) |_{J^\sigma})$$

for $z \in \mathfrak{f}$. Once again we see that $\xi^\sigma \in \mathcal{F}$. Moreover, since $J^\sigma = \mathcal{G}'(\chi_{S'})_{\sigma 1}$ by (42), we have $\text{ad}(\ell'_s) |_{J^\sigma} = \sigma \text{id}_{J^\sigma}$, so $\xi^\sigma(\ell'_s) = \sigma \dim(J^\sigma) \neq 0$.

Now, since \mathcal{F} is one dimensional, we have $\nu^\sigma = r^\sigma \xi^\sigma$ for some $r^\sigma \in \mathbb{K}$, in which case (evaluating at ℓ'_s) we have $r^\sigma = \frac{\mu^+(\ell'_s)}{\dim(J^\sigma)}$ for $\sigma = \pm$. So for $z \in \mathfrak{f}$ we have

$$\nu^\sigma(z) = \frac{\mu^+(\ell'_s)}{\dim(J^\sigma)} \text{tr}(\text{ad}(z) |_{J^\sigma}).$$

But if $x \in J^\sigma$, $a \in J^{-\sigma}$, we have $[x, a] \in \mathfrak{f}$ since $\mathcal{G}'(\chi_{S'})$ is 3-graded. Also $\tau^\sigma(x, a) = \nu^\sigma([x, a])$ by (43), so

$$\tau^\sigma(x, a) = \frac{\mu^+(\ell'_s)}{\dim(J^\sigma)} \text{tr}(D_J^\sigma(x, a)).$$

Finally, we compute $\mu^+(\ell'_s)$. Let $C = C(\Pi)$ with (i, j) -entry $c_{ij} := \langle \mu_i, \mu_j \rangle$, and let $\{\ell_i\}$ be the basis for \mathcal{H} that is dual to $\{\mu_i\}$. Then

$$h_j = \sum_i c_{ij} \ell_i$$

for $1 \leq j \leq n$. Similarly, since $C(\Pi')$ has (i, j) -entry c_{ij} for $i, j \neq t$, we have $h_j = \sum_{i \neq t} c_{ij} \ell_i$ for $j \neq t$. So letting $D' = C(\Pi')^{-1}$ with (i, j) -entry d'_{ij} for $i, j \neq t$, we have $\ell'_j = \sum_{i \neq t} d'_{ij} h_i$ for $j \neq t$. Therefore

$$\begin{aligned} \ell'_s &= \sum_{i \neq t} d'_{is} h_i = \sum_{i \neq t} d'_{is} \sum_k c_{ki} \ell_k = \sum_k \left(\sum_{i \neq t} c_{ki} d'_{is} \right) \ell_k \\ &= \left(\sum_{i \neq t} c_{ti} d'_{is} \right) \ell_t + \sum_{k \neq t} \left(\sum_{i \neq t} c_{ki} d'_{is} \right) \ell_k = c_{tt'} d'_{t's} \ell_t + \ell_s, \end{aligned}$$

since $c_{ti} = 0$ for $i \neq t'$. Thus $\mu^+(\ell'_s) = c_{tt'} d'_{t's} \mu^+(\ell_t) + \mu^+(\ell_s) = \theta_{\Pi, T}$. \square

Theorem 9.2.2 shows that the reflection \check{P} of any Kantor pair P satisfying the given assumptions can be constructed in the form $\mathfrak{T}(J, \tau)$, where J and τ are described in terms of marked Dynkin diagrams. In the next corollary, we describe J and τ using classical matrix constructions.

Our notation in Columns 3 and 4 of Table 1 follows [L, §17.4]. Indeed, $I_{p,q}$ is the Jordan pair $(M_{p,q}(\mathbb{K}), M_{p,q}(\mathbb{K}))$ with products $\{x, a, y\}^\sigma = xa^t y + ya^t x$; Π_n is the subpair $(A_n(\mathbb{K}), A_n(\mathbb{K}))$ of $I_{n,n}$, where $A_n(\mathbb{K})$ is the space of alternating $n \times n$ -matrices; IV_n is the Jordan pair $(\mathbb{K}^n, \mathbb{K}^n)$ with products $\{x, a, y\}^\sigma = q(x, a)y + q(y, a)x - q(x, y)a$, where $q : \mathbb{K}^n \times \mathbb{K}^n \rightarrow \mathbb{K}$ is the non-degenerate symmetric bilinear form on \mathbb{K}^n ; and V is the Jordan pair $(M_{1,2}(\mathcal{C}), M_{1,2}(\mathcal{C}))$ with products $\{x, a, y\}^\sigma = x(\bar{a}^t y) + y(\bar{a}^t x)$, where \mathcal{C} is the (split) Cayley algebra with standard involution $c \mapsto \bar{c}$ and trace form $t_{\mathcal{C}} : \mathcal{C} \rightarrow \mathbb{K}$ given by $t_{\mathcal{C}}(c) = c + \bar{c}$ [L, §12.10].

Corollary 9.2.3. *Suppose we have the assumptions and notation of Theorem 9.2.2. The first column of Table 1 lists the possibilities, up to diagram automorphism, for the marked Dynkin diagrams representing $P = \mathfrak{P}(\Pi; S, T)$ (with the restriction on the rank n of Π also indicated). The second column lists the corresponding marked Dynkin diagram representing $\check{P} \simeq_{gr} \mathfrak{P}(\Pi; \sigma_{\Pi'}(S'), T)$ (see Corollary 8.3.6). Finally, in each row, we have*

$$\check{P} \simeq_{gr} \mathfrak{T}(J, \tau),$$

where J and τ are listed in Columns 3 and 4 of the table respectively.

Proof. Constructing Column 1 is an easy exercise considering types case-by-case using the discussion at the beginning of this subsection.

To complete row i , where $1 \leq i \leq 7$, we chose S and Π as listed in Column 1. To get the entry in Column 2, we just calculate $\sigma_{\Pi'}(S')$. To get the entries in Columns 3 and 4, we do the following:

- (a) Find a Jordan pair J of matrices that is isomorphic to $\mathfrak{P}(\Pi'; S')$; and then
- (b) Calculate $\text{tr}(D_J^\sigma(x, a))$ for $x \in J^\sigma$, $a \in J^{-\sigma}$ (which allows us to calculate τ^σ using (40)).

Fortunately, (a) and (b) can be accomplished using well known facts from the theory of Jordan pairs. Indeed for (a), Loos and Neher have written down a table which lists the finite dimensional simple Jordan pairs (as pairs of matrices) together with their representing marked Dynkin diagrams. (See [LN, §19.9], which refers to [N1]

$P = \mathfrak{P}(\Pi; S, T)$	n	$\check{P} \simeq_{\text{gr}} \mathfrak{P}(\Pi; \sigma_{\Pi'}(S'), T)$	J	$\tau^\sigma(x, a)$
	≥ 2		$I_{1,n-1}$	xa^t
	≥ 3		IV_{2n-3}	$q(x, a)$
	≥ 2		$I_{1,n-1}$	$2xa^t$
	≥ 4		IV_{2n-4}	$q(x, a)$
	≥ 5		$I_{2,n-2}$	$\text{tr}(xa^t)$
	6		Π_5	$\frac{1}{2} \text{tr}(xa)$
	7		V	$\text{t}_C(x\bar{a}^t)$

TABLE 1. Reflections of simple close-to-Jordan pairs (with a case excluded)

and [N2] for the necessary arguments.) For (b), Meyberg observes in [M2, (2.20)] (using Theorem 17.3 in [L]) that

$$\text{tr}(D_J^+(x, a)) = g_J m_J(x, a) \quad (47)$$

for $x \in J^+$, $a \in J^-$, where g_J is the genus of J and $m_J : J^+ \times J^- \rightarrow \mathbb{K}$ is the generic trace of J .

We carry out steps (a) and (b) in detail for the second last row of the table, leaving the other rows to the reader. In that row the marked diagram representing $\mathfrak{P}(\Pi'; S')$ is isomorphic to . So from the table in [LN, §19.9], we see that we may take $J = \Pi_5 = (A_5(\mathbb{K}), A_5(\mathbb{K}))$. Then from [L, §17.2] we know that $g_J = 8$ and $m_J(x, a) = \frac{1}{2} \text{tr}(xa)$, which using (47) gives us the equality $\text{tr}(D^+(x, a)) = 4 \text{tr}(xa)$. So since $J = J^{\text{op}}$, we have $\text{tr}(D^\sigma(x, a)) = 4 \text{tr}(xa)$ for $\sigma = \pm$. Thus, since $\dim(J^\sigma) = 10$, we have by (40) that $\tau^\sigma(x, a) = \frac{2}{5} \theta_{\Pi, T} \text{tr}(xa)$. Now, labeling the roots in Π as in Example 5.3.1, we have $t = 1$, $t' = 2$ and $s = 6$. Also, the $(1, 2)$ -entry of $C(\Pi)$ is -1 , and the $(2, 6)$ -entry of $C(\Pi')^{-1}$ is $\frac{3}{4}$ [H, Table 1, §13.2], so $\theta_{\Pi, T} = -\frac{3}{4} + 2 = \frac{5}{4}$. Hence $\tau^\sigma(x, a) = \frac{1}{2} \text{tr}(xa)$. \square

Note that if we ignore the grading, the ungraded Kantor pair \check{P} in the second last row of Table 1 is the pair labelled $E_6(20, 5)$ in Example 5.3.1. Our construction of $E_6(20, 5)$ in the form $\mathfrak{T}(P, \tau)$ is a basis-free pair version, with full proofs, of the construction given by Kantor in [K1, §6.6] (see also [K2, §4]) of the Kantor triple system C_{55}^2 . More precisely, $\mathfrak{T}(J, \tau)$ shown in the table is the double of C_{55}^2 . The pair $E_6(20, 5)$ is of particular interest since it is one of only two finite dimensional simple Kantor pairs of exceptional type that does not arise by doubling a structurable algebra. (See [AFS, §7.9], where another construction of $E_6(20, 5)$

is given as the reflection of an SP-graded Kantor pair that is constructed using exterior algebras.)

9.3. The excluded case. In this final subsection, we consider, without proofs, the case that was excluded in Theorem 9.2.2 and Corollary 9.2.3. We make the assumptions and use the notation of the first three paragraphs of Subsection 9.2.

To treat the excluded case, we assume that $\Pi = \{\mu_1, \dots, \mu_n\}$ is of type A_n with roots labelled in order on the diagram from left to right, $n \geq 3$, $S = \{\mu_1, \mu_n\}$ and $T = \{\mu_t\}$ with $1 < t < n$. Let $P = \mathfrak{P}(\Pi; S, T)$, in which case the marked Dynkin diagrams representing P and $\check{P} \simeq_{\text{gr}} \mathfrak{P}(\Pi; \sigma_{T'}(S'), T)$ are respectively:

$$\odot \cdots \bullet \cdots \bullet \cdots \odot \quad \text{and} \quad \cdots \odot \cdots \bullet \cdots \bullet \cdots$$

Note that Π' is not connected (which is the reason we are treating this case separately). In fact, the connected components of Π' are $\Pi'_1 = \{\mu_1, \dots, \mu_{t-1}\}$ and $\Pi'_2 = \{\mu_{t+1}, \dots, \mu_n\}$. One sees as in Theorem 9.2.2(i) that $\mathfrak{P}(\Pi'_1; \{\mu_1\})$ and $\mathfrak{P}(\Pi'_2; \{\mu_n\})$ are Jordan pairs with

$$P_0 = \mathfrak{P}(\Pi'_1; \{\mu_1\}) \oplus \mathfrak{P}(\Pi'_2; \{\mu_n\}).$$

Next let $J = J_1 \oplus J_2$, where $J_1 \simeq \mathfrak{P}(\Pi'_1; \{\mu_1\})$ and $J_2 \simeq \mathfrak{P}(\Pi'_2; \{\mu_n\})$ are simple Jordan pairs. We obtain as in Theorem 9.2.2(ii) that $\check{P} \simeq_{\text{gr}} \mathfrak{T}(J, \tau)$, where

$$\tau^\sigma(x_1 + x_2, a_1 + a_2) = \frac{1}{t} \text{tr}(D_{J_1}^\sigma(x_1, a_1)) + \frac{1}{n-t+1} \text{tr}(D_{J_2}^\sigma(x_2, a_2)) \quad (48)$$

for $x_i \in J_i^\sigma$, $a_i \in J_i^{-\sigma}$.

Finally, we may choose

$$J = J_1 \oplus J_2, \quad \text{where } J_1 = I_{1,t-1} \text{ and } J_2 = I_{1,n-t}; \quad (49)$$

and we see as in Corollary 9.2.3 that

$$\check{P} \simeq_{\text{gr}} \mathfrak{T}(J, \tau) \quad \text{with} \quad \tau^\sigma(x_1 + x_2, a_1 + a_2) = x_1 a_1^t + x_2 a_2^t$$

(In this last equation, the superscript t denotes the transpose map.)

Remark 9.3.1. The proof of the above facts is obtained by modifying the arguments in Subsection 9.2. The main difference is in the proof of (48), where we consider two different spaces of homomorphisms \mathcal{F}_1 and \mathcal{F}_2 obtained from simple 3-graded Lie algebras which envelop the Jordan pairs $\mathfrak{P}(\Pi'_1; \{\mu_1\})$ and $\mathfrak{P}(\Pi'_2; \{\mu_n\})$ respectively (just as \mathcal{F} is obtained from the simple 3-graded Lie algebra $\mathcal{G}'(\chi_{S'})$ in the proof of (40)). We leave the details to the reader.

Remark 9.3.2. The reader may have noticed that the Jordan pairs J that appear in either Column 3 of Table 1 or in (49) consist of all finite dimensional semi-simple Jordan pairs of rank 1 or 2 [L, §17]; and that τ is almost uniquely determined by J . The intrinsic reason for these mysterious facts will be explained in [AF2], where we will use Jordan techniques to study the construction $\mathfrak{T}(J, \tau)$ over an arbitrary field of characteristic $\neq 2$ or 3.

REFERENCES

- [ABG] B. Allison, G. Benkart and Y. Gao, *Lie algebras graded by the root systems* BC_r , $r \geq 2$, Mem. Amer. Math. Soc. **158** (2002), no. 751.
- [AF1] B. Allison and J. Faulkner, *Elementary groups and invertibility for Kantor pairs*, Comm. Algebra **27** (1999), 519-556.

- [AF2] B. Allison and J. Faulkner, *Constructing Kantor pairs as tensor products: A Jordan theoretic approach*, in preparation.
- [AFS] B. Allison, J. Faulkner and O. Smirnov, *Weyl images of Kantor pairs*, to appear in the *Canad. J. of Math.*, arXiv:1404.3339v2.
- [At] K. Atsuyama, *On the algebraic structures of graded Lie algebras of second order*, *Kodai Math. J.* **5** (1982), 225-229.
- [BS] G. Benkart and O. Smirnov, *Lie algebras graded by the root system BC_1* , *J. Lie Theory* **13** (2003), 91-132.
- [B] N. Bourbaki, *Lie groups and Lie algebras, Chapters 4-6 and 7-9*, Translated from the French, *Elements of Mathematics*, Springer-Verlag, Berlin, 2002 and 2005.
- [E1] A. Elduque, *New simple Lie superalgebras in characteristic 3*, *J. Algebra* **296** (2006), 196-233.
- [E2] A. Elduque, *The magic square and symmetric compositions II*, *Rev. Mat. Iberoam.* **23** (2007), 57-84.
- [EKO] A. Elduque, N. Kamiya, A. Okubo, *$(-1, -1)$ -balanced Freudenthal Kantor triple systems and noncommutative Jordan algebras*, *J. Algebra* **294** (2005), 19-40.
- [EK] A. Elduque and M. Kochetov, *Gradings on simple Lie algebras*, *Mathematical Surveys and Monographs*, 189, American Mathematical Society, Providence, RI; Atlantic Association for Research in the Mathematical Sciences (AARMS), Halifax, NS, 2013.
- [F] J. Faulkner, *A construction of Lie algebras from a class of ternary algebras*. *Trans. Amer. Math. Soc.* **155** (1971), 397-408.
- [FF] J. Faulkner and J. Ferrar, *On the structure of symplectic ternary algebras*, *Nederl. Akad. Wetensch. Proc. Ser. A 75=Indag. Math.* **34** (1972), 247-256.
- [GLN] E. García, M. Gómez Lozano and E. Neher, *Nondegeneracy for Lie triple systems and Kantor pairs*, *Canad. Math. Bull.* **54** (2011), 442-455.
- [H] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer-Verlag, New York, 1972.
- [Kac] V. Kac, *Infinite dimensional Lie algebras*, third edition, Cambridge University Press, Cambridge, 1990.
- [K1] I.L. Kantor, *Some generalizations of Jordan algebras* (Russian), *Trudy Sem. Vektor. Tenzor. Anal.* **13** (1972), 407-499.
- [K2] I.L. Kantor, *Models of exceptional Lie algebras*, *Soviet Math. Dokl.* **14** (1973), 254-258.
- [KS] I.L. Kantor and I.M. Skopec, *Freudenthal trilinear operations* (Russian), *Trudy. Sem. Vektor. Tenzor. Anal.* **18** (1978), 250-263.
- [L] O. Loos, *Jordan pairs*, *Lecture Notes in Mathematics* **460**, Springer-Verlag, Berlin, 1975.
- [LN] O. Loos and E. Neher, *Locally finite root systems*, *Mem. Amer. Math. Soc.* **171** (2004).
- [Mc] K. McCrimmon, *A taste of Jordan algebras*, Universitext, Springer-Verlag, New York, 2004.
- [M1] K. Meyberg, *Eine Theorie der Freudenthalschen Tripelsysteme I, II*, *Nederl. Akad. Wetensch. Proc. Ser. A 71=Indag. Math.* **30** (1968) 162-174, 175-190.
- [M2] K. Meyberg, *Trace formulas and derivations in simple Jordan pairs*, *Comm. Algebra* **12** (1984), 1311-1326.
- [N1] E. Neher, *Lie algebras graded by 3-graded root systems and Jordan pairs covered by grids*, *Amer. J. Math.* **118** (1996), 439-491.
- [N2] E. Neher, *Quadratic Jordan superpairs covered by grids*, *J. Algebra* **269** (2003), 28-73.
- [YA] K. Yamaguti and H. Asano, *On the Freudenthal's construction of exceptional Lie algebras*, *Proc. Japan Acad.* **51** (1975), 253-258.
- [YO] K. Yamaguti and A. Ono, *On representations of Freudenthal-Kantor triple systems $U(\epsilon, \delta)$* , *Bull. Fac. School Ed. Hiroshima Univ. Part II* **7** (1984), 43-51.

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